

Overconvergent family of Siegel-Hilbert modular forms

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Abstract

We construct one parameter families of overconvergent Siegel-Hilbert modular forms. In particular, for any classical Siegel-Hilbert modular eigenform one can find a rigid analytic disc centered at this point, on which an infinite family of classical points with varying weights accumulates at the center.

We use the strategy of Kisin-Lai [KL], with the new ingredients being the canonical integral model of Pappas-Rapoport [PR05] and the compactifications of Shimura varieties in the ramified cases of Lan [La12].

This result plays an important role in the construction of Galois representations for GL_2 over CM fields in [Mo11].

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1 Introduction

The study of p -adic families of automorphic forms has been carried out in many works. In the case of elliptic modular forms, the overconvergent modular eigenforms of *finite slope* (i.e. with non-zero Hecke eigenvalue at p) are interpolated to be points on a rigid analytic curve, which is known as the Coleman-Mazur eigencurve [CM98]. Before this seminal work, the slope zero case (i.e. family of ordinary eigenforms) was obtained by Hida [Hi86].

Among all the approaches to the construction of eigenvarieties for more general algebraic groups, the work of Kisin-Lai [KL] on overconvergent Hilbert modular forms is most closely related to ours. Their method is a generalization of that of Coleman-Mazur. In both cases, the key point for interpolating modular forms is to show the *complete continuity* (cf. P. 425 [Co97] for definition) of the Atkin-Lehner operator on certain spaces of overconvergent forms, which Kisin-Lai obtained by a study of overconvergence of the canonical subgroup in the Hilbert modular case via formal geometry (theory of Raynaud). There is differences between the approach in [CM98] and that in [KL]. In the case of elliptic modular forms, one has the result of Deligne on Eisenstein series, which is of level one, of weight $p - 1$ and of slope zero, whose reduction mod p is the Hasse invariant. This allows Coleman-Mazur to interpolate modular forms of varying weights by twisting by p -adic analytic families of Eisenstein series. However, in the more general (Siegel-)Hilbert modular case the slope zero result on Eisenstein series is not yet available. Instead, one lifts (a certain power of) the Hasse invariant in characteristic p to be a global section of certain automorphic line bundle over the integral model of the Shimura variety.

We would like to mention certain differences between our method and that of [KL], which are mainly caused by the generality of the Siegel-Hilbert moduli space.

Firstly, in both [KL] and our case, one needs certain good properties of the integral Shimura variety as well as its compactifications, in order to lift the Hasse invariant. In the Hilbert modular case of [KL], they glue the toroidal compactification of the Rapoport model [Ra78] with the Deligne-Pappas model [DP], because the Rapoport model may not be proper at the places which are ramified in the totally real field F . Fortunately, Rapoport's toroidal compactification can be used because the Lie algebra condition, which causes non-properness at finite distance, is automatic in the boundary. In the Siegel-Hilbert case, one has to do more to take care of the ramified places. There exists the canonical integral model of Pappas-Rapoport [PR05] in the Siegel-Hilbert modular case, which has a moduli interpretation. Its toroidal compactifications are, however, not completely understood (cf. [La12] for some results along this direction). Fortunately, the (partial) toroidal compactifications and minimal compactification for the ordinary locus is constructed in [La12] successfully, which will be enough for our use.

Furthermore, we follow the idea of Hida [Hi02] to form the formal Igusa tower over the ordinary part of the formal completion of the (compactified) moduli space with level structure away from p , instead of using the “unramified $\Gamma_0(Np^n)$ cusps” in [KL]. This seems more convenient in the general Siegel-Hilbert case.

Finally, we refer to the result in [Fa] for the overconvergence of the canonical subgroups in the p -power torsions of abelian schemes, which is the most complete among the works on canonical subgroups.

With these strategies and results, we construct one dimensional families of modular eigenforms on $\mathrm{GSp}_{2g/F}$ for any totally real field F and $g \geq 1$. More precisely, we obtain, for each classical weight κ for $\mathrm{GSp}_{2g/F}$, a reduced rigid analytic curve \mathcal{E}_κ , whose points are in one-to-one correspondence with systems of Hecke eigenvalues of overconvergent Siegel-Hilbert modular forms, whose weights “differ” from that of κ by parallel weights. One of the key properties of the rigid curve \mathcal{E}_κ is that the canonical map to the weight space given by the weights of the modular forms is, locally in the domain, finite flat. We refer the reader to Theorem 4.13 for more details.

Essentially due to the (local) finite flatness of the weight map on \mathcal{E}_κ , we have the following result for the accumulation of classical points in the curve \mathcal{E}_κ .

Theorem 1.1 (Theorem 4.15). *Let f be a (classical) Siegel-Hilbert modular eigenform of weight κ and with level p^n at p . Let k_0 be any sufficiently big integer which is not divided by p .*

For any positive integers t with large enough p -adic valuation, there exist Siegel-Hilbert modular eigenforms f_t of the same level and of weight $\kappa \cdot \mathrm{Nm}^{(p-1)k_0 t}$, whose Hecke eigenvalues converge p -adically to that of f when t goes to zero p -adically.

One would like to replace Theorem 1.1 by the Zariski density of classical points on \mathcal{E}_κ , provided that one had an analogue of the Coleman inequality in the Siegel-Hilbert case asserting the classicity of overconvergent forms (of integral weight) of small slopes. However, this theorem can still be useful in applications. For example, Theorem 1.1 is one of the main ingredients for attaching Galois representations to automorphic forms π on GL_2 over arbitrary CM fields, the work of one of us [Mo11]. More precisely, in order to construct such a 2-dimensional representation, one first lifts π to an automorphic form Π (of non-cohomological type!) on

$\mathrm{GSp}_4(\mathbf{A}_F)$. On the other hand, the Galois representations for automorphic forms of cohomological type in the $\mathrm{GSp}_4(F)$ case are constructed in [Mo11]. Then the Galois representation ρ_Π associated to Π is obtained by interpolating the Galois representations associated to forms on $\mathrm{GSp}_4(\mathbf{A}_F)$ of cohomological type, with the family of cohomological forms being supplied by Theorem 1.1. As is mentioned in [Mo11], the use of p -adic analytic family of automorphic forms, compared to the use of congruence relations between them, has the advantage that this (less elementary) method allows us to prove local-global compatibility for ρ_Π at places of F not dividing p , up to semi-simplification.

We would like to mention that the whole eigenvariety for Siegel modular forms (not necessarily with parallel weights) was recently developed in [AIP], under the assumption $F = \mathbf{Q}$. Our result, on the other hand, applies to all totally real fields and allows any ramification of p in F .

The paper is organized as follows.

In Section 2, we recall the results on integral models of PEL Shimura varieties and their compactifications from [La12]. The minimal compactification will allow us to lift the Hasse invariant in positive characteristic to characteristic zero.

In the next section, we recall the notion of relative compactness and systems of overconvergent neighborhoods, as well as the result on overconvergence of canonical subgroups of [Fa]. At the end of Section 3, we use the idea of Hida to form the formal Igusa tower.

In the last section, we form the spaces of overconvergent Siegel-Hilbert modular eigenforms and the Hecke operators on them. Then we prove that the $U_{(p)}$ -operator related to the overconvergent canonical subgroup is completely continuous on the spaces above. Based on these facts and the machinery in [CM98], we construct the rigid curve interpolating these overconvergent forms. Finally, by the finite flatness of the weight map, we are able to show Theorem 1.1.

Notation

- F is a totally real field of degree d over \mathbf{Q} with difference \mathcal{D} , and $\mathcal{O} = \mathcal{O}_F$ is the ring of integers. We denote by $\mathbf{A} = \mathbf{A}_F$ the ring of adeles of F , and by \mathbf{A}_f the ring of finite adeles of F .
- $p \geq 2$ is a fixed rational prime.
- If K/\mathbf{Q}_p is a finite extension, K_0 is the maximal unramified extension of \mathbf{Q}_p in K , and $[K : K_0] = e$, $[K_0 : \mathbf{Q}_p] = f$.
- $\bar{\mathbf{Q}}_p$ is a fixed algebraic closure of K , and \mathbf{C}_p is the completion of $\bar{\mathbf{Q}}_p$ for the p -adic topology.
- For a maximal torus $T = T_G$ of a reductive group G over \mathbf{Z} , $\mathrm{Nm} : \mathrm{Res}_{\mathbf{Z}}^{\mathcal{O}} T \rightarrow T$ is the norm map, i.e. for any ring R , $\mathrm{Nm}(R) : T(\mathcal{O} \otimes_{\mathbf{Z}} R) \rightarrow T(R)$ is given by the norm $N_{F/\mathbf{Q}}$ on F .
- If $C \subset G(\mathbf{A}_f)$ denotes an open compact subgroup, then we write it in the form $C = C_p C^p$, where $C_p \subset G(\mathbf{Q}_p)$, $C^p \subset G(\mathbf{A}_f^p)$, and \mathbf{A}_f^p denotes the ring of finite adeles over F with trivial p -component.

- If \mathfrak{X} is a formal scheme, then $\mathfrak{X}^{\text{rig}}$ denotes the associated rigid space.
- For the sheaf ω^κ on schemes (resp. rigid spaces), we use the same symbols to denote their pull-backs and extensions via the canonical maps.

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2 Siegel-Hilbert moduli spaces

2.1 PEL datum

2.1.1 The general integral PEL data

We recall the (integral) PEL datum $(\mathcal{O}_B, *, L, \psi, h)$, whose rational part $(B, *, L_{\mathbf{Q}} := L \otimes_{\mathbf{Z}} \mathbf{Q}, \psi \otimes_{\mathbf{Z}} \mathbf{Q}, h)$ can give rise to a Shimura datum by 4.1 [Ko92].

- B is a finite dimensional semisimple \mathbf{Q} -algebra whose center is denoted by F , and is equipped with a positive involution $*$:

$$(ab)^* = b^* a^*, b^{**} = b, \quad \forall a, b \in B,$$

$$\text{Tr}_{B/\mathbf{Q}}(bb^*) > 0, \quad \forall b \neq 0.$$

\mathcal{O}_B is an order of B stabilized by the involution above.

- (L, ψ) is a symplectic $(\mathcal{O}_B, *)$ -module over \mathbf{Z} , i.e. L is a finite free \mathbf{Z} -module carrying an alternating form $\psi : L \times L \rightarrow \mathbf{Z}$, such that

$$\psi(bx, y) = \psi(x, b^*y), \quad \forall x, y \in L, b \in \mathcal{O}_B.$$

Let G be the group over \mathbf{Z} so that for any \mathbf{Z} -algebra R ,

$$G(R) = \{g \in \text{GL}_{(\mathcal{O}_B)_R}(L_R) \mid \psi(gx, gy) = c(g)\psi(x, y), c(g) \in R\}.$$

- Let

$$\tilde{h} : \mathbf{C} \rightarrow \text{End}_{(\mathcal{O}_B)_{\mathbf{R}}}(L_{\mathbf{R}})$$

be an \mathbf{R} -algebra homomorphism that gives a Hodge structure of type $(1, 0), (0, 1)$ on $L_{\mathbf{R}}$, such that $\psi(x, \tilde{h}(\sqrt{-1})y)$ is a symmetric positive definite bilinear form on $L_{\mathbf{R}}$. The restriction $\tilde{h}|_{\mathbf{C}^\times}$ can be viewed as a morphism of \mathbf{R} -algebraic groups

$$h : \text{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m, \mathbf{C}} \longrightarrow G_{\mathbf{R}}.$$

The action of h gives a decomposition

$$L_{\mathbf{C}} = V_{0,\mathbf{C}} \oplus V_{1,\mathbf{C}} \quad (2.1.1)$$

where h acts on the first factor by the character $z \mapsto \bar{z}$ and on the second one by $z \mapsto z$. The Shimura field is then by definition

$$E = F[\mathrm{Tr}_{\mathbf{C}}(b|V_{0,\mathbf{C}}), b \in B].$$

The decomposition (2.1.1) is then defined over the subfield E of \mathbf{C} :

$$L_E = V_0 \oplus V_1. \quad (2.1.2)$$

2.1.2 PEL data for symplectic groups

Let $B = F$ be a totally real field of degree d . Let $\mathcal{O}_B = \mathcal{O}_F = \mathcal{O}$ and $*$ = Id be the trivial involution. Let L be a finite free \mathbf{Z} -module of rank $2dg$ equipped with an \mathcal{O} -module structure, together with the standard symplectic form

$$\varphi : L \times L \rightarrow \mathcal{O}$$

given by the antisymmetric matrix $J = \begin{pmatrix} & -I_{dg} \\ I_{dg} & \end{pmatrix}$. Set

$$\psi = \mathrm{Tr}_{\mathcal{O}/\mathbf{Z}} \circ \varphi.$$

The \mathbf{C} -algebra homomorphism \tilde{h} is

$$a + bi \mapsto \begin{pmatrix} aI_{dg} & -bI_{dg} \\ bI_{dg} & aI_{dg} \end{pmatrix}.$$

We have the PEL datum $(\mathcal{O}, \mathrm{Id}, L, \psi, h = \tilde{h}|_{\mathbf{C}^\times})$. In this case

$$G = \mathrm{Res}_{\mathcal{O}/\mathbf{Z}} \mathrm{GSp}_{2g},$$

where GSp_{2g} is the split reductive group of symplectic similitudes respecting the matrix J .

The Shimura field in this case is $E = \mathbf{Q}$. This fact guarantees the nonemptiness of the ordinary locus of the Shimura variety corresponding to the Shimura datum above.

2.2 The Siegel-Hilbert moduli space over the Shimura field

Keep the Shimura datum $(\mathcal{O}, \mathrm{Id}, L, \psi, h)$ as above. Let $H \subset G(\hat{\mathbf{Z}})$ be an open compact subgroup. We recall the moduli problem from Section 5 [Ko92] and 1.4.1.4 [La08].

Let \mathcal{M}_H be the functor that assigns to a \mathbf{Q} -scheme S the isomorphism classes of the tuples $(A, i, \lambda, \alpha_H)$ of the following kind

- A is an abelian scheme over S of relative dimension dg , equipped with an \mathcal{O} -action, called the *real multiplication*: $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$.

- The requirement of Kottwitz determinant condition

$$\det_{\mathcal{O}_S}(b|\mathrm{Lie} A) = \det_E(b|V_0), \quad \forall b \in F$$

as *polynomial functions*, for which both sides of the equality are considered as morphisms of S -schemes. (cf. Section 5 [Ko92] or 3.23 [RZ] for details.)

- $\lambda : A \rightarrow A^\vee$ is a polarization. Recall that a symmetric homomorphism $A \rightarrow A^\vee$ is a *polarization* if (locally for the étale topology) it comes from a line bundle over A which is ample over S . (cf. 6.2 [GIT].)
- α_H is an H -level structure of type $(L_{\mathbf{Z}}, \psi)$ as defined in 1.3.7.6 [La08].

The functor \mathcal{M}_H is represented by a separated smooth algebraic stack of finite type over $E = \mathbf{Q}$, by Artin's theory and Grothendieck's theory of Hilbert schemes. If H is neat, then it is represented by a smooth quasi-projective (not necessarily proper) scheme over \mathbf{Q} , by [GIT] (and the theory of Hilbert schemes). We refer the reader to [La08] for details.

We denote the universal abelian scheme over \mathcal{M}_H by \mathcal{A} , and denote by ω by the pull-back along the unit section of the relative differentials $\Omega_{\mathcal{A}/\mathcal{M}_H}^1$.

Remark 2.1. *Let X denote the $G(\mathbf{R})$ -conjugacy classes of \tilde{h} . The complex manifold $G(\mathbf{Q}) \backslash X \times G(\mathbf{A}^\infty)/H$ descends to a quasi-projective scheme Sh_H over \mathbf{Q} , which is commonly called the Shimura variety. We have a canonical open and closed immersion*

$$Sh_H \hookrightarrow [\mathcal{M}_H]$$

of the Shimura variety into the coarse moduli space of the algebraic stack \mathcal{M}_H .

As a special case of the construction of \mathcal{M}_H , we have the functor \mathcal{M}_{H^p} with the level structure α_H being the prime to p level structure on H^p .

2.3 Integral model of Siegel-Hilbert moduli space and compactifications

In Chapter 4 [La12], Lan constructs a normal and flat algebraic stack $\vec{\mathcal{M}}_H$ over $\mathbf{Z}_{(p)}$ which comes with a canonical isomorphism

$$\vec{\mathcal{M}}_H \times_{\mathrm{Spec} \mathbf{Z}_{(p)}} \mathrm{Spec} \mathbf{Q} \simeq \mathcal{M}_H.$$

We recall the construction briefly.

One first finds an auxiliary Shimura datum which can provide the canonical integral model and toroidal compactification. In fact, one can embed the \mathbf{Z} -module L into another finite free \mathbf{Z} -module L_{aux} which comes with an alternating pairing ψ_{aux} whose restriction to L is ψ . The \mathbf{R} -algebra homomorphism \tilde{h} then induces another \mathbf{R} -algebra homomorphism \tilde{h}_{aux} , whose restriction to \mathbf{C}^\times is denoted by h_{aux} . Moreover, we have a subring $\mathcal{O}_{\mathrm{aux}} \subset \mathcal{O}$ for which the embedding $L \hookrightarrow L_{\mathrm{aux}}$ is $\mathcal{O}_{\mathrm{aux}}$ -linear. The point is that, for the auxiliary Shimura datum $(\mathcal{O}_{\mathrm{aux}}, \mathrm{Id}, L_{\mathrm{aux}}, \psi_{\mathrm{aux}}, h_{\mathrm{aux}})$, the prime p is a *good* prime to which the main results of [La08] apply.

Now we have an induced homomorphism of algebraic groups over \mathbf{Z}

$$t : G \longrightarrow G_{\text{aux}}$$

where the second group is defined by the auxiliary Shimura datum in the same way as before. The auxiliary Shimura datum provides a moduli stack $\mathcal{M}_{G_{\text{aux}}(\hat{\mathbf{Z}}^p)}$ which is separated smooth and of finite type over $\mathbf{Z}_{(p)}$.

By the fact that p is a good prime for $\mathcal{M}_{G_{\text{aux}}(\hat{\mathbf{Z}}^p)}$, one can show that there is a canonical isomorphism

$$\mathcal{M}_{G_{\text{aux}}(\hat{\mathbf{Z}}^p) \times G_{\text{aux}}(\mathbf{Z}_p)} \simeq \mathcal{M}_{G_{\text{aux}}(\hat{\mathbf{Z}}^p)} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

More generally, for any open compact subgroup $H_{\text{aux}} = H_{\text{aux}}^p G_{\text{aux}}(\mathbf{Z}_p) \subset G_{\text{aux}}(\hat{\mathbf{Z}})$ such that H^p is mapped to H_{aux}^p under the morphism $t : G(\mathbf{Z}^p) \rightarrow G_{\text{aux}}(\mathbf{Z}^p)$, we have similarly a moduli stack $\mathcal{M}_{H_{\text{aux}}^p}$ for which p is a good prime and a morphism

$$\mathcal{M}_H \rightarrow \mathcal{M}_{H_{\text{aux}}^p} \otimes_{\mathbf{Z}} \mathbf{Q}, \quad (2.3.1)$$

compatible with the map between the two PEL data, which is finite on the coarse moduli spaces. Moreover, for a subgroup $H'_{\text{aux}} = H_{\text{aux}}^p G_{\text{aux}}(\mathbf{Z}_p) \subset H_{\text{aux}}$ the morphism (2.3.1) is recovered by the composite of $\mathcal{M}_H \rightarrow \mathcal{M}_{H'_{\text{aux}}^p} \otimes_{\mathbf{Z}} \mathbf{Q}$ with the natural map $\mathcal{M}_{H'_{\text{aux}}^p} \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \mathcal{M}_{H_{\text{aux}}^p} \otimes_{\mathbf{Z}} \mathbf{Q}$.

Proposition 2.2. *The normalization $\vec{\mathcal{M}}_H$ of $\mathcal{M}_{H_{\text{aux}}^p}$ in \mathcal{M}_H is a normal flat algebraic stack over $\mathbf{Z}_{(p)}$ whose generic fibre is canonically isomorphic to \mathcal{M}_H .*

The normalization of $[\mathcal{M}_{H_{\text{aux}}^p}]$ in $[\mathcal{M}_H]$ under the map of coarse moduli spaces induced by (2.3.1) is canonically isomorphic to $[\vec{\mathcal{M}}_H]$, which is a quasi-projective normal flat scheme over $\mathbf{Z}_{(p)}$. Hence $\vec{\mathcal{M}}_H \simeq [\vec{\mathcal{M}}_H]$ is a scheme if H is neat.

Let $\mathcal{M}_H^{\text{tor}}$ be the toroidal compactification of \mathcal{M}_H for a fixed admissible smooth rational polyhedral cone decomposition datum Σ for \mathcal{M}_H .

Proposition 2.3. (1) *There is an admissible smooth rational polyhedral cone decomposition datum Σ_{aux} for $\mathcal{M}_{H_{\text{aux}}^p}$ (hence the toroidal compactification $\mathcal{M}_{H_{\text{aux}}^p}^{\text{tor}}$ of $\mathcal{M}_{H_{\text{aux}}^p}$), which is compatible with Σ in a natural way, and induces a canonical morphism*

$$\mathcal{M}_H^{\text{tor}} \longrightarrow \mathcal{M}_{H_{\text{aux}}^p}^{\text{tor}} \otimes_{\mathbf{Z}} \mathbf{Q}, \quad (2.3.2)$$

which is compatible with the stratifications on both sides (in particular, extending (2.3.1)) and the pull-back of universal objects.

(2) *Let \mathcal{M}_H^{\min} and $\mathcal{M}_{H_{\text{aux}}^p}^{\min}$ be the corresponding minimal compactifications, and let $(\det \omega)^{k_0}$ and $(\det \omega)_{\text{aux}}^{k_0}$ be the k_0 -th power of the determinant of (the pull-back along the unit section of) the relative differentials of the universal semi-abelian scheme \mathcal{A} over the toroidal compactifications $\mathcal{M}_H^{\text{tor}}$ and $\mathcal{M}_{H_{\text{aux}}^p}^{\text{tor}}$, respectively. Here we fix a sufficiently large k_0 , which can be taken to be 1 if H is neat. Then the morphism (2.3.2) induces a natural morphism*

$$\mathcal{M}_H^{\min} \longrightarrow \mathcal{M}_{H_{\text{aux}}^p}^{\min} \otimes_{\mathbf{Z}} \mathbf{Q}, \quad (2.3.3)$$

which is compatible with the stratifications on both sides, and for any integer $a_0 \geq 1$ the pull-back of $(\det \omega)_{\text{aux}}^{a_0 k_0}$ is canonically isomorphic to $(\det \omega)^{a k_0}$ for some integer $a \geq a_0$ determined by the choice of the auxiliary \mathbf{Z} -module L_{aux} .

(3) The normalization $\vec{\mathcal{M}}_H^{\min}$ of $\mathcal{M}_{H_{\text{aux}}^p}^{\min}$ in \mathcal{M}_H^{\min} is a projective normal flat scheme over $\mathbf{Z}_{(p)}$ whose generic fibre is canonically isomorphic to \mathcal{M}_H^{\min} . It contains $\vec{\mathcal{M}}_H$ as an open dense subscheme. The line bundle $(\det \omega)^{a k_0}$ extends to an ample line bundle over $\vec{\mathcal{M}}_H^{\min}$ which is canonically isomorphic to the pull-back of $(\det \omega)_{\text{aux}}^{a_0 k_0}$.

(4) In the case that H is neat and Σ is projective, there is an integral model $\vec{\mathcal{M}}_H^{\text{tor}}$ for the toroidal compactification $\mathcal{M}_H^{\text{tor}}$, which is by construction the normalization of the blowup of certain coherent ideal sheaf on $\vec{\mathcal{M}}_H^{\min}$. It is a project normal flat scheme over $\mathbf{Z}_{(p)}$, such that $\vec{\mathcal{M}}_H^{\text{tor}} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q} \simeq \mathcal{M}_H^{\text{tor}}$ in a canonical way. If $H' \subset H$ is an open compact subgroup, then there is a canonical map $\vec{\mathcal{M}}_{H'}^{\text{tor}} \rightarrow \vec{\mathcal{M}}_H^{\text{tor}}$, compatible with the canonical map $\mathcal{M}_{H'} \rightarrow \mathcal{M}_H$.

For the integral model $\vec{\mathcal{M}}_{H^p}$ with prime to p level, we have the following stronger result:

Theorem 2.4 ([PR05]). *The canonical map $\vec{\mathcal{M}}_{H^p} \rightarrow \mathcal{M}_{H_{\text{aux}}^p}$ is a closed embedding.*

In particular, we have a moduli interpretation for $\vec{\mathcal{M}}_{H^p}$, with PEL data as part of the moduli problem.

Proof. By Theorem 12.2 of [PR05], the flat scheme theoretic image in $\vec{\mathcal{M}}_{H_{\text{aux}}^p}$ of the generic fibre \mathcal{M}_{H^p} is normal, hence is canonically isomorphic to $\vec{\mathcal{M}}_{H^p}$. In another word, the integral model $\vec{\mathcal{M}}_{H^p}$ defined above coincides with the canonical integral model of Pappas-Rapoport [PR05].

For the last claim, the reader is referred to Section 15 [PR05] for detail. □

2.4 Ordinary locus and its partial compactifications

Let S be a scheme over $\mathbf{Z}_{(p)}$.

2.4.1 Level structures prime to p

Let A be an abelian scheme over S , equipped with a polarization λ and \mathcal{O} -endomorphism i as before. Let $H^p \subset G(\hat{\mathbf{Z}}^p)$ be an open compact. Let N be a natural number prime to p such that $H^p \supset U(N)$, the principal mod N congruence subgroup. A principal level N structure of (A, λ, i) of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$ is the pair (α_N, ν_N) defined as follows:

- $\alpha_N : L/NL \xrightarrow{\sim} A[N]$ is an \mathcal{O} -linear isomorphism of group schemes over S , such that

(1) the symplectic pairing

$$L/NL \times L/NL \longrightarrow \mathbf{Z}/N\mathbf{Z}$$

and the λ -Weil pairing

$$A[N] \times A[N] \longrightarrow \mu_N$$

induced by the polarization λ are compatible for a chosen isomorphism of group schemes

$$\nu_N : \mathbf{Z}/N\mathbf{Z} \xrightarrow{\sim} \mu_N$$

with respect to a fixed primitive N -th root of unity ζ_N .

(2) α_H is symplectic liftable: there is a tower of finite étale surjections

$$(S_M \twoheadrightarrow S_N = S)_{N|M, p \nmid M}$$

and \mathcal{O} -linear isomorphisms $\alpha_M : L/ML \xrightarrow{\sim} A[M]$ with respect to an isomorphism $\nu_M : \mathbf{Z}/M\mathbf{Z} \xrightarrow{\sim} \mu_M$ such that for any valid indices $M'|M''$

$$(\alpha_{M'}, \nu_{M'}) = (\alpha_{M''}, \nu_{M''}) \bmod M'.$$

(This condition is required so that α_N lifts, at any geometric point s of S , to an \mathcal{O} -linear symplectic isomorphism between $L_{\mathbf{Z}^p}$ and the Tate module of A_s .)

Consider all natural numbers such that $p \nmid N$ and $H^p \supset U(N)$. A level H^p structure of (A, λ, i) of type $(L_{\mathbf{Z}^p}, \psi)$ is a collection of $H^p/U(N)$ -orbits of principal level N structures (α_N, ν_N) for all N as above.

2.4.2 Ordinary level structures at p

Let

$$0 = D^1 \subset D^0 \subset D^{-1} = L_{\mathbf{Z}_p}$$

be a filtration of $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p$ -modules, such that $\mathrm{Gr}_D^{-1} := D^{-1}/D^0$ is torsion free as a \mathbf{Z}_p -module, and under the pairing ψ D^0 is totally isotropic and is its own annihilator. Such a filtration determines a filtration

$$0 = D^{\vee,1} \subset D^{\vee,0} \subset D^{\vee,-1} = L_{\mathbf{Z}_p}^{\vee}$$

on the dual lattice $L_{\mathbf{Z}_p}^{\vee}$. We have the natural map

$$\varphi_D^0 : D^0 \rightarrow D^{\vee,0}$$

between the 0-th graded pieces, whose reduction mod p^n by φ_{D,p^n}^0 , for any $n \in \mathbf{Z}_{\geq 0}$.

Let $P_D \subset G_{\mathbf{Z}_p}$ be the subgroup over \mathbf{Z}_p , which is the stabilizer of D . Let M_D be the group over \mathbf{Z}_p , whose R -points, for any \mathbf{Z}_p -algebra R , are $(g, c) \in \mathrm{GL}_{\mathcal{O} \otimes_{\mathbf{Z}} R}(\mathrm{Gr}_D \otimes_{\mathbf{Z}_p} R) \times \mathbf{G}_m(R)$ such that $\psi(gx, gy) = c\psi(x, y)$. We denote by U_D the kernel of the natural morphism from P_D to M_D . Now for any integer $n \in \mathbf{Z}_{\geq 0}$, we set

$$U_{p,0}(p^n) = (G(\mathbf{Z}_p) \rightarrow G(\mathbf{Z}/p^n\mathbf{Z}))^{-1} P_D(\mathbf{Z}/p^n\mathbf{Z}),$$

$$U_{p,1}^{\mathrm{bal}}(p^n) = (G(\mathbf{Z}_p) \rightarrow G(\mathbf{Z}/p^n\mathbf{Z}))^{-1} U_D(\mathbf{Z}/p^n\mathbf{Z}).$$

Let S be a scheme over \mathbf{Z} . Let A be an abelian scheme together with a polarization λ and an \mathcal{O} -endomorphism i . An ordinary principal level p^n structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$ is the following data:

- An \mathcal{O} -linear homomorphism of group schemes over S , $\alpha_{p^n}^0 : (D^0/p^n D^0)^{\text{mult}} \rightarrow A[p^n]$.
- An \mathcal{O} -linear homomorphism of group schemes over S , $\alpha_{p^n}^{\vee,0} : (D^{\vee,0}/p^n D^{\vee,0})^{\text{mult}} \rightarrow A^\vee[p^n]$.
- A section ν_{p^n} of $(\mathbf{Z}/p^n \mathbf{Z})^\times \simeq \text{Isom}_S(\mu_{p^n}, \mu_{p^n})$ so that the homomorphism of multiplicative group schemes $\nu_{p^n} \circ (\varphi_{D,p^n}^0)^{\text{mult}} : (D^0/p^n D^0)^{\text{mult}} \rightarrow (D^{\vee,0}/p^n D^{\vee,0})^{\text{mult}}$ is compatible with λ under $\alpha_{p^n}^0$ and $\alpha_{p^n}^{\vee,0}$, and such that the images of $\alpha_{p^n}^0$ and $\alpha_{p^n}^{\vee,0}$ kill each other under the λ -Weil paring on $A[p^n] \times A^\vee[p^n]$.
- The requirement that α_{p^n} is symplectic liftable: there is a tower of quasi-finite étale surjections

$$(S_{p^{n'}} \twoheadrightarrow S_{p^n} = S)_{n' \geq n}$$

and triples $(\alpha_{p^{n'}}^0, \alpha_{p^{n'}}^{\vee,0}, \nu_{p^{n'}})$ as above such that for any $n'' \geq n'$,

$$(\alpha_{p^{n''}}^0, \alpha_{p^{n''}}^{\vee,0}, \nu_{p^{n''}}) \bmod p^{n'} = (\alpha_{p^{n'}}^0, \alpha_{p^{n'}}^{\vee,0}, \nu_{p^{n'}}).$$

Let $H_p \subset G(\mathbf{Z}_p)$ be an open compact subgroup such that $U_{p,1}^{\text{bal}}(p^n) \subset H_p \subset U_{p,0}(p^n)$ for some integer $n \geq 0$. An ordinary level H_p structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$ is an $H_p/U_{p,1}^{\text{bal}}(p^n)$ -orbit of ordinary principal level p^n structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$.

2.4.3 Integral model with ordinary level structure

Let $H = H^p H_p \subset G(\hat{\mathbf{Z}})$ be an open compact subgroup such that $U_{p,1}^{\text{bal}}(p^n) \subset H_p \subset U_{p,0}(p^n)$ for some integer $n \geq 0$.

Let $\mathcal{M}_H^{\text{ord,naive}}$ be the functor that assigns to a $\mathbf{Z}_{(p)}$ -scheme S the isomorphism classes of the tuples $(A, i, \lambda, \alpha_{H^p}, \alpha_{H_p})$ as follows:

- A is an abelian scheme over S of relative dimension dg , equipped with an \mathcal{O} -endomorphism $i : \mathcal{O} \hookrightarrow \text{End}_S(A)$.
- $\lambda : A \rightarrow A^\vee$ is a polarization.
- α_{H^p} is an H^p -level structure of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$.
- α_{H_p} is an H_p -level structure of type $(L_{\mathbf{Z}_p}, \psi, D)$.

The functor $\mathcal{M}_H^{\text{ord,naive}}$ is represented by an algebraic stack of finite type over $\mathbf{Z}_{(p)}$. We refer the reader to [La12] for details.

Let r_H be the fixed nonnegative integer as in Section 8 [La12] which is determined by H and the filtration D of $L_{\mathbf{Z}_p}$. One can check that, over any $\mathbf{Q}[\zeta_{p^{r_H}}]$ -scheme S , there is a natural assignment from the level H structures of $(A, i, \lambda)_S$ to the pairs of H^p -level structure of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$ and H_p -level structure of type $(L_{\mathbf{Z}_p}, \psi)$, which is in fact injective. As a consequence, we have an open and closed immersion

$$\mathcal{M}_H \otimes_{\mathbf{Z}} \mathbf{Q}[\zeta_{p^{r_H}}] \longrightarrow \mathcal{M}_H^{\text{ord,naive}} \otimes_{\mathbf{Z}} \mathbf{Q}[\zeta_{p^{r_H}}],$$

whose image is denoted by M_H^{ord} , which is an open and closed algebraic substack of $\mathcal{M}_H^{\text{ord,naive}} \otimes_{\mathbf{Z}} \mathbf{Q}[\zeta_{p^rH}]$.

Proposition 2.5. *The normalization $\vec{\mathcal{M}}_H^{\text{ord}}$ of $\mathcal{M}_H^{\text{ord,naive}}$ in M_H^{ord} under the natural morphism $M_H^{\text{ord}} \rightarrow \mathcal{M}_H^{\text{ord,naive}}$ is a regular algebraic stack, which is separated smooth and of finite type over $\mathbf{Z}_{(p)}[\zeta_{p^rH}]$. We denote the pull-back to $\vec{\mathcal{M}}_H^{\text{ord}}$ of the universal tuple on $\mathcal{M}_H^{\text{ord,naive}}$ by the same symbol and call it the universal tuple on $\vec{\mathcal{M}}_H^{\text{ord}}$.*

There is a canonical morphism $\vec{\mathcal{M}}_H^{\text{ord}} \rightarrow \vec{\mathcal{M}}_H$ which is an open immersion on the coarse moduli spaces. If H is neat, then $\vec{\mathcal{M}}_H^{\text{ord}}$ is a quasi-projective scheme over $\mathbf{Z}_{(p)}[\zeta_{p^rH}]$, which is an open subscheme of $\vec{\mathcal{M}}_H$.

2.4.4 Partial compactifications of the ordinary locus

Keep the data as before.

Theorem 2.6. *There is an algebraic stack $\vec{\mathcal{M}}_H^{\text{ord,tor}}$, separated smooth and of finite type over $\mathbf{Z}_{(p)}[\zeta_{p^rH}]$, containing $\vec{\mathcal{M}}_H^{\text{ord}}$ as an open dense algebraic substack. In the case that H is neat, $\vec{\mathcal{M}}_H^{\text{ord,tor}}$ is a quasi-projective scheme over $\mathbf{Z}_{(p)}[\zeta_{p^rH}]$.*

The universal tuple $(\mathcal{A}, i, \lambda, \alpha_{H^p}, \alpha_{H_p})$ extends to $\vec{\mathcal{M}}_H^{\text{ord,tor}}$. The boundary $\vec{\mathcal{M}}_H^{\text{ord,tor}} \setminus \vec{\mathcal{M}}_H^{\text{ord}}$ is a relative Cartier divisor with normal crossing.

We have the Hodge line bundle $(\det \omega)_{\vec{\mathcal{M}}_H^{\text{ord}}}$ over $\vec{\mathcal{M}}_H^{\text{ord}}$. Let $(\det \omega)_{\vec{\mathcal{M}}_H^{\text{ord,tor}}}$ be the Hodge line bundle given by the universal semi-abelian scheme over $\vec{\mathcal{M}}_H^{\text{ord,tor}}$. We form

$$\vec{\mathcal{M}}_H^{\text{ord,min}} = \text{Proj}(\oplus_{k \geq 0}) \Gamma(\vec{\mathcal{M}}_H^{\text{ord,tor}}, (\det \omega)_{\vec{\mathcal{M}}_H^{\text{ord,tor}}}^k).$$

This is in general not projective, as the partial toroidal compactification $\vec{\mathcal{M}}_H^{\text{ord,tor}}$ is in general not proper.

Theorem 2.7. *There exists a canonical proper morphism*

$$\vec{\mathcal{M}}_H^{\text{ord,tor}} \longrightarrow \vec{\mathcal{M}}_H^{\text{ord,min}},$$

and $\vec{\mathcal{M}}_H^{\text{ord,min}}$ is quasi-projective, being an open subscheme of the projective scheme $\vec{\mathcal{M}}_H^{\text{min}} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q}[\zeta_{p^rH}]$.

The scheme $\vec{\mathcal{M}}_H^{\text{ord,min}}$ is normal flat over $\mathbf{Z}_{(p)}[\zeta_{p^rH}]$ which contains $[\vec{\mathcal{M}}_H^{\text{ord}}]$ as an open dense subscheme. There is a minimal natural number $k_0 \geq 1$ such that the line bundle $(\det \omega)_{\vec{\mathcal{M}}_H^{\text{ord}}}^{k_0}$ is the pull-back of an ample line bundle over $\vec{\mathcal{M}}_H^{\text{ord,min}}$. In the case that H is neat, we can take $k_0 = 1$.

Remark 2.8. *For the moduli space $\vec{\mathcal{M}}_{H^p}$ with prime to p level, the integral model $\vec{\mathcal{M}}_{H^p}^{\text{ord}}$ is simply the ordinary locus of $\vec{\mathcal{M}}_{H^p}$. It then comes with a moduli interpretation, by Theorem 2.4.*

We write the image of the proper morphism $\vec{\mathcal{M}}_H^{\text{ord,tor}} \longrightarrow \vec{\mathcal{M}}_H^{\text{ord,min}}$ as $\vec{\mathcal{M}}_H^{\text{ord,*}}$.

From now on, we always assume H is neat.

2.5 Hecke operators and geometric correspondences

Through the end of this section, let \mathcal{M} denote $\mathcal{M}_{H^p}, \mathcal{M}_H, \mathcal{M}_H^{\text{ord}}, \mathcal{M}_H^{\text{ord, naive}}, \mathcal{M}_{H_{\text{aux}}^p}$, or $\vec{\mathcal{M}}_{H^p}$. In the last case, $\vec{\mathcal{M}}_{H^p}$ will only be considered as the moduli space of the PEL data, which will be enough for our purpose.

2.5.1 The double-coset Hecke algebra

Let q be a prime number and $v \mid q$ a place in F . For the completion F_v of F at the place v , we denote by \mathcal{O}_v the integer ring and fix a uniformizer ϖ_v and denote by \mathcal{O}_v the integer ring of F_v . We define the spherical Hecke algebra $\mathcal{H}_v^{\text{sph}}$ for $\text{GSp}_{2g}(F_v)$ with coefficients in \mathbf{Z} to be the algebra of \mathbf{Z} -valued functions on $\text{GSp}_{2g}(F_v)$ that are bi-invariant under $\text{GSp}_{2g}(\mathcal{O}_v)$. It is generated by the characteristic functions on the following double cosets:

$$T_{v,1} = \text{GSp}_{2g}(\mathcal{O}_v) \begin{pmatrix} I_g & & & \\ & \varpi_v I_g & & \\ & & & \\ & & & \end{pmatrix} \text{GSp}_{2g}(\mathcal{O}_v),$$

$$T_{v,i} = \text{GSp}_{2g}(\mathcal{O}_v) \begin{pmatrix} I_{g-i+1} & & & \\ & \varpi_v I_{i-1} & & \\ & & \varpi_v^2 I_{g-i+1} & \\ & & & \varpi_v I_{i-1} \end{pmatrix} \text{GSp}_{2g}(\mathcal{O}_v), \quad 2 \leq i \leq g,$$

and

$$S_v = \varpi_v \text{GSp}_{2g}(\mathcal{O}_v).$$

2.5.2 Weights and automorphic vector bundles

Let $T_{g/\mathbf{Z}}$ be the standard diagonal maximal torus of $\text{GSp}_{2g/\mathbf{Z}}$. Put $G/\mathbf{Z} = \text{Res}_{\mathbf{Z}}^{\mathcal{O}} \text{GSp}_{2g}$ and $T = \text{Res}_{\mathbf{Z}}^{\mathcal{O}} T_g$. Take the standard Borel B of G with unipotent radical U and identify $T = B/U$. We work over a ring (resp. sheaf of rings over a scheme) R . Let M be the Levi of the standard Siegel parabolic of G . Then $M \supset T$.

Consider a character

$$\kappa : T = \text{Res}_{\mathbf{Z}}^{\mathcal{O}} T_g \rightarrow \mathbf{G}_m.$$

We may regard κ as a character of $B \cap M$ which is trivial on $U \cap M$. The character κ is called *dominant* with respect to B , if the induced representation $\text{Ind}_{B \cap M}^M \kappa^{-1}$ inside the rational functions of the scheme $(M/U \cap M)_R$ is non-zero.

The Bruhat-Tits decomposition shows that the subspace $(\text{Ind}_{B \cap M}^M \kappa^{-1})^{U \cap M}$ is one dimensional, and T acts on a generator by $-w_0 \kappa$, where w_0 is the longest element in the Weyl group (with respect to T). The M -translation of the generator generates a sub-representation

$$\rho_{\kappa}^{-1} \subset \text{Ind}_{B \cap M}^M \kappa^{-1},$$

where an element m in the standard Levi M acts as $m \cdot f(x) = f(m^{-1}x)$. The R -dual ρ_{κ} of ρ_{κ}^{-1} is called the *rational representation of highest weight κ* , which has the universal property

that for any M -module X ,

$$\mathrm{Hom}_M(\rho_\kappa, X) \simeq \mathrm{Hom}_M(X^*, \rho_\kappa^{-1}) \simeq \mathrm{Hom}_B(X^*, -\kappa) \simeq \mathrm{Hom}_B(\kappa, X).$$

We define the *automorphic vector bundle* on \mathcal{M} to be

$$\omega^\kappa = \mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^g, \omega) \times^L \rho_\kappa$$

to be the contracted product as in Section 2 [Hi02]. The sections of ω^κ are functions on $\mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^g, \omega)$ such that

$$f(\varphi m) = \rho_\kappa(m^{-1})f(\varphi), \quad \forall \varphi \in \mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^g, \omega), m \in M,$$

where we keep in mind that the Levi M acts on the sheaf ω via ρ_κ . (Note that ω^κ is written as $\rho_\kappa(\omega)$ in [Hi02].)

We define the space of (classical) Siegel-Hilbert modular forms of weight κ (of level the level of \mathcal{M}) with coefficients in a ring R , to be the global sections

$$H^0(\mathcal{M}_R, \omega^\kappa).$$

2.5.3 Geometric correspondences

Let \mathfrak{a} be an ideal of \mathcal{O} . Let $\mathcal{M}^\mathfrak{a}$ be the moduli stack of isogenies between objects in \mathcal{M} , that is, the algebraic stack representing the functor $\mathcal{M}^\mathfrak{a}$ which assigns to any base scheme S over \mathbf{Q} (resp. $\mathbf{Z}_{(p)}$) the category in groupoids in which an object is an isogeny

$$f : A \rightarrow B$$

between two polarized abelian schemes with level structure and endomorphism $(A, i_A, \lambda_A, (\alpha_H)_A)$ and $(B, i_B, \lambda_B, (\alpha_H)_B)$, whose kernel is \mathcal{O} -linearly isomorphic to $(\mathcal{O}/\mathfrak{a}\mathcal{O})^g$ and intersects with (the image of) the level structure only along the unit section, compatible with the \mathcal{O} -endomorphisms, and respects the polarizations on both sides.

Here we obtain the representability of the functor $\mathcal{M}^\mathfrak{a}$ by the use of the fact that \mathcal{M} is representable and by the theory of Hilbert schemes (cf. P. 251 [FC]). In particular, for H neat, the functor $\mathcal{M}^\mathfrak{a}$ is represented by a quasi-projective scheme over \mathbf{Q} (resp. $\mathbf{Z}_{(p)}$), which is denoted by the same symbol, as usual. The universal isogeny over $\mathcal{M}^\mathfrak{a}$ is denoted by $\mathcal{I}^\mathfrak{a}$.

Assigning such an isogeny to its source (resp. target), we have two natural projections

$$\mathcal{M} \xleftarrow{\pi_{1,\mathfrak{a}}} \mathcal{M}^\mathfrak{a} \xrightarrow{\pi_{2,\mathfrak{a}}} \mathcal{M},$$

whose restrictions to any connected component Z of $\mathcal{M}^\mathfrak{a}$ are proper, by the valuative criterion.

A useful fact is that when p is invertible in the base scheme S , the two projections

$$\pi_{i,(p)} : \mathcal{M}^{(p)} \rightarrow \mathcal{M}$$

are finite étale.

Assume q is a prime invertible in the base scheme S . For $v \mid q$ be a primer ideal in \mathcal{O} , one

has the bijection between the connected components of \mathcal{M}^v and the double cosets γ_v in the spherical Hecke algebra $\mathcal{H}_v^{\text{sph}}$. Denote the corresponding connected component of \mathcal{M}^v by \mathcal{M}^{γ_v} , over which the universal isogeny is said to be of type γ_v . We have the two projections

$$\pi_{i,\gamma_v} : \mathcal{M}^{\gamma_v} \rightarrow \mathcal{M}, \quad i = 1, 2,$$

of type γ_v ,

We will use the same symbols $\pi_{\mathfrak{a}}, \pi_{i,\gamma_v}$ for the induced maps on the associated formal scheme and rigid spaces that appear later. Sometimes we simply use the notation π_i when non confusion arises.

2.5.4 Hecke operators

Let R be an algebra over \mathbf{Q} or \mathbf{F}_p .

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{I}^{\mathfrak{a}} & \longrightarrow & \mathcal{A} \\ f_{\mathfrak{a}} \downarrow & & \downarrow f \\ Z = \mathcal{M}^{\mathfrak{a}} & \xrightarrow{\pi_{i,\mathfrak{a}}} & \mathcal{M}. \end{array}$$

Over the base $S = \text{Spec } R$, we have a natural map of $\mathcal{O} \otimes_{\mathbf{Z}} \mathcal{O}_{Z_R}^{\times}$ -torsors

$$\pi_{2,\mathfrak{a}}^* \omega = \pi_{2,\mathfrak{a}}^* (f_* \Omega_{\mathcal{A}/\mathcal{M}}^1) \rightarrow f_{\mathfrak{a}*} \Omega_{\mathcal{I}^{\mathfrak{a}}/Z}^1 \xrightarrow{\sim} f_{\mathfrak{a}*} (\pi_{1,\mathfrak{a}}^* \Omega_{\mathcal{A}/\mathcal{M}}^1) \xrightarrow{\sim} \pi_{1,\mathfrak{a}}^* \omega,$$

hence the induced map

$$\theta : \pi_{2,\mathfrak{a}}^* \omega^{\kappa} \longrightarrow \pi_{1,\mathfrak{a}}^* \omega^{\kappa}.$$

Applying $\pi_{1,\mathfrak{a}*}$ and composing with the trace map

$$\text{tr} : \pi_{1,\mathfrak{a}*} \pi_{1,\mathfrak{a}}^* \omega^{\kappa} \rightarrow \omega^{\kappa},$$

we obtain the map

$$\pi_{1,\mathfrak{a}*} \pi_{2,\mathfrak{a}}^* \omega^{\kappa} \xrightarrow{\pi_{1,\mathfrak{a}*} \theta} \pi_{1,\mathfrak{a}*} \pi_{1,\mathfrak{a}}^* \omega^{\kappa} \xrightarrow{\text{tr}} \omega^{\kappa}.$$

Composing the natural map

$$\omega^{\kappa} \rightarrow \pi_{1,\mathfrak{a}*} \pi_{2,\mathfrak{a}}^* \omega^{\kappa}$$

with the one above and taking global sections, one gets the desired endomorphism

$$T_{\mathfrak{a}} : H^0(\mathcal{M}_R, \omega^{\kappa}) \rightarrow H^0(\mathcal{M}_R, \omega^{\kappa}),$$

which will be denoted by $U_{(p)}$ in the case $\mathfrak{a} = (p)$. We remark that the Hecke operator $U_{(p)}$ corresponds to the product of $T_{v,1}, v|p$.

We have the same construction for $Z = \mathcal{M}^{\gamma_v}$, the connected component of \mathcal{M}^v of type γ_v with v invertible in R . In these cases, the Hecke operators corresponding to the double cosets $T_{v,i}$ (resp. S_v) will be denoted by $T_{v,i}$ (resp. S_v) again.

2.6 Hasse invariants and liftings

2.6.1 Hasse invariants on abelian schemes in characteristic p

Let R be an \mathbf{F}_p -algebra and A a semi-abelian scheme of rank n equipped with a polarization. We fix a basis $\mathbf{v} = \{v_1, \dots, v_n\}$ of the $\mathcal{O} \otimes_{\mathbf{Z}} R$ -module $H^0(A, \Omega_{A/R}^1)$. By 7.2 [Ka70], there is a map of $\mathcal{O}_{A^{(p)}}$ -modules, which is called the *Cartier operator*

$$C_{A/R} : (\mathrm{Frob}_{A/R})_* \Omega_{A/R}^n \rightarrow \mathrm{Frob}_R^* \Omega_{A/R}^n,$$

where $A^{(p)}$ is the fibre product of the structure map $A \rightarrow \mathrm{Spec} R$ and the absolute Frobenius Frob_R on $\mathrm{Spec} R$, and $\mathrm{Frob}_{A/R} : A \rightarrow A^{(p)}$ is the relative Frobenius. The Cartier operator induces the Frobenius on the Lie algebra:

$$\begin{aligned} C_{A/R} : H^0(A, \Omega_{A/R}^n) &\xrightarrow{\sim} H^0(A^{(p)}, (\mathrm{Frob}_{A/R})_* \Omega_{A/R}^n) \\ &\rightarrow H^0(A, \mathrm{Frob}_R^* \Omega_{A/R}^n) \xrightarrow{\sim} \mathrm{Frob}_R^* H^0(A, \Omega_{A/R}^n), \end{aligned}$$

which shows that there is a constant $h(A, \lambda, \mathbf{v}) \in R$, such that

$$C_{A/R}(\det \mathbf{v}) = h(A, \lambda, \mathbf{v}) \mathrm{Frob}_R^*(\det \mathbf{v}).$$

We note that $h(A, \lambda, \mathbf{v})$ is independent of the choice of the polarization and if we change the basis \mathbf{v} to another one $M\mathbf{v}$ by an invertible matrix M , then

$$h(A, \lambda, M\mathbf{v}) = \det^{1-p}(M) h(A, \lambda, \mathbf{v}).$$

Applying this to the special fibre of $\mathcal{M} = \vec{\mathcal{M}}_H^{\mathrm{ord}, \mathrm{tor}}$ and the universal tuple on it, we then have a global section

$$h \in H^0(\mathcal{M}_{\mathbf{F}_p}, (\det \omega)^{\otimes p-1}),$$

which is known as the Hasse invariant of the moduli space.

Remark 2.9. For R an \mathbf{F}_p -algebra and A an abelian scheme over R , the Hasse invariant $h(A)$ is non-vanishing if and only if A is ordinary. (cf. the argument on P. 24 [Hi02])

Lemma 2.10. The natural map of sheaves on $(\mathcal{M}_H^{\mathrm{a}, \mathrm{ord}, \mathrm{naive}})_{\mathbf{F}_p}$ (resp. $(\mathcal{M}_H^{\gamma_v, \mathrm{ord}, \mathrm{naive}})_{\mathbf{F}_p}$ when v is away from p)

$$\theta : \pi_2^*(\det \omega)^{\otimes p-1} \rightarrow \pi_1^*(\det \omega)^{\otimes p-1}$$

satisfies

$$\theta(\pi_2^* h) = \pi_1^* h.$$

Proof. This is by the functoriality of the Cartier operator.

Let R be an \mathbf{F}_p -algebra. Note that

$$\theta(\pi_2^* h)(A \rightarrow B, \mathbf{v}, \mathbf{v}') = h(B, \mathbf{v}'), \quad (\pi_1^* h)(A \rightarrow B, \mathbf{v}, \mathbf{v}') = h(A, \mathbf{v}),$$

where \mathbf{v}' is a chosen basis of $H^0(B, \Omega_{B/R}^1)$ whose pull-back via π_2 is \mathbf{v} . On the other hand, we

have

$$\begin{aligned} h(B, \mathbf{v}') \text{Frob}_R^*(\det \mathbf{v}) &= h(B, \mathbf{v}') \text{Frob}_R^*(\pi_2^*(\det \mathbf{v}')) \\ &= (\pi_2^{(p)})^* \mathcal{C}_{B/R}(\det \mathbf{v}') = \mathcal{C}_{A/R}(\pi_2^*(\det \mathbf{v}')), \end{aligned}$$

where the last equality is obtained by the étaleness of the projection π_2 , and we have written $\pi_2^{(p)} = \text{Frob}_R^* \pi_2$.

Now the result follows, as by definition

$$\mathcal{C}_{A/R}(\pi_2^*(\det \mathbf{v}')) = \mathcal{C}_{A/R}(\det \mathbf{v}) = h(A, \mathbf{v}) \text{Frob}_R^*(\det \mathbf{v}).$$

□

2.6.2 Lifting Hasse invariants to characteristic zero

Recall that $(\vec{\mathcal{M}}_H^{\text{ord}, \min})_{\mathbf{F}_p}$ is normal and the complement of $(\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{F}_p}$ in it has codimension at least 2. Then any section of the power of Hodge line bundle $(\det \omega)^{\otimes k}$ on $(\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{F}_p}$ extends to $(\vec{\mathcal{M}}_H^{\text{ord}, \min})_{\mathbf{F}_p}$. In particular, the Hasse invariant h extends uniquely to a section in

$$H^0((\vec{\mathcal{M}}_H^{\text{ord}, \min})_{\mathbf{F}_p}, (\det \omega)^{\otimes p-1}),$$

which we denote by the same symbol. Since the line bundle $(\det \omega)^{\otimes p-1}$ on $(\vec{\mathcal{M}}_H^{\text{ord}, \min})_{\mathbf{F}_p}$ is ample, the line bundle $(\det \omega)^{\otimes k(p-1)}$ is very ample for any sufficiently large integer k . Hence h^k lifts to a section in $H^0((\vec{\mathcal{M}}_H^{\text{ord}, \min})_{\mathbf{Z}_p}, (\det \omega)^{\otimes (p-1)k})$. Restricting this section to $(\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{Z}_p}$, we get

Proposition 2.11. *For $k \in \mathbf{Z}_{\geq 0}$ big enough, the section $h^k \in H^0((\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{F}_p}, (\det \omega)^{\otimes (p-1)k})$ lifts to a section*

$$\widetilde{h^k} \in H^0((\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{Z}_p}, (\det \omega)^{\otimes (p-1)k}).$$

Proof. It is standard to show by Serre vanishing that

$$H^1(\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{Z}_p}, (\det \omega)^{\otimes (p-1)k} = 0$$

when k is sufficiently large, which in turns gives the surjectivity of

$$H^0((\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{Z}_p}, (\det \omega)^{\otimes (p-1)k}) \rightarrow H^0((\vec{\mathcal{M}}_H^{\text{ord}})_{\mathbf{F}_p}, (\det \omega)^{\otimes (p-1)k}).$$

We refer to 2.3.5.1 [Ti06] for more details. □

From now on, we fix such a lift as in Proposition 2.11

$$E = \widetilde{h^{k_0}}$$

such that $k \gg 0$ and $p \nmid k_0$.

3 Analitification of Siegel-Hilbert moduli schemes

3.1 Preliminary

We recall certain definitions and results from [KL] and [Lü], which will be applied to the rigid analytifications of the Siegel-Hilbert moduli schemes, as well as their formal models and the (pull-back of) automorphic vector bundles.

3.1.1 Relative compactness

Let K be a finite extension of \mathbf{Q}_p and \mathcal{O}_K the ring of integers. For \mathfrak{X} a formal scheme over $\mathrm{Spf} \mathcal{O}_K$, we denote by $\mathfrak{X}^{\mathrm{rig}}$ the rigid analytic space associated to it, and by \mathfrak{X}_0 its special fibre. For $\mathfrak{U} \subset \mathfrak{X}$ an admissible open, we let $]\mathfrak{U}_0[$ denote the tube of \mathfrak{U} , i.e. the pre-image in $\mathfrak{X}^{\mathrm{rig}}$ of the \mathfrak{U}_0 under the natural specialization $\mathfrak{X}^{\mathrm{rig}} \rightarrow \mathfrak{X}_0$, which is surjective. If $f : X \rightarrow Y$ is morphism of rigid spaces, we call a morphism of ϖ -adic \mathcal{O}_K -flat formal schemes $\tilde{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$ a formal model of f , if f is the rigidification of \tilde{f} .

Definition 3.1. Let $f : X \rightarrow V$ be a quasi-compact morphism of rigid analytic spaces, and $U \subset X$ a quasi-compact (relative to X) admissible open. We say U is *relatively compact in X over V* , denoted by

$$U \Subset_V X,$$

if there exists an admissible covering of quasi-compact subsets $\{V_i\}$ of V such that there exists for each $f|_{V_i}$ a formal model $\tilde{f}_i : \mathfrak{X}_i \rightarrow \mathfrak{Y}_i$ and an open subset $\mathfrak{U}_i \subset \mathfrak{X}_i$ giving $U|_{V_i} =](\mathfrak{U}_i)_0[$, and such that the closure in $(\mathfrak{X}_i)_0$ of $(\mathfrak{U}_i)_0$ is proper over $(\mathfrak{Y}_i)_0$. Equivalently, this means locally in V there is a closed V -immersion of X into an n -dimensional unit ball $\mathbf{D}_V^n(1)$ under which U maps into a ball $\mathbf{D}^n(\epsilon)$ for some $\epsilon < 1$.

If $V = \mathrm{Sp} K$, we simply write $U \Subset_V X$ as $U \Subset X$.

The notion of relative compactness in Definition 3.1 is independent of the choice of covering $\{V_i\}$, essentially by Raynaud's theorem that the category of quasi-compact rigid spaces is equivalent to that of quasi-compact admissible formal schemes localized by admissible blow ups. (cf. 2.1.2 [KL] for more details.)

Lemma 3.2 (2.1.8 [KL]). *Let $i : Y \hookrightarrow Y'$ and $j : X \hookrightarrow X'$ be admissible open inclusions, all of which are quasi-compact rigid spaces over the quasi-compact rigid space V , and $f : Y' \rightarrow X'$ be a proper morphism. Suppose we have the following Cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow i & & \downarrow j \\ Y' & \xrightarrow{f} & X' \end{array}$$

and suppose $X \Subset_V X'$. Then $Y \Subset_V Y'$.

3.1.2 Systems of neighborhoods of a rigid space

Let $X \subset \bar{X}$ be an admissible open quasi-compact subset of the rigid space \bar{X} . We define the collection

$$X^\dagger = \{\text{admissible open quasi-compact subsets } X' \subset \bar{X} \text{ such that } X \Subset_{\bar{X}} X'\}.$$

For $Y \subset \bar{Y}$ another such pair, we can define the set of (equivalence classes of) morphisms between X^\dagger and Y^\dagger :

$$\text{Hom}(Y^\dagger, X^\dagger) = \{f : Y' \rightarrow X' \mid Y' \in Y^\dagger, X' \in X^\dagger, \text{ and } f(Y) \subset X\} / \sim,$$

where \sim is the smallest equivalence relation generated by the relation that $(f_1 : Y_1 \rightarrow X_1) \sim (f_2 : Y_2 \rightarrow X_2)$ if $X_1 \subset X_2$, $Y_1 \subset Y_2$ and $f_2|_{Y_1} = f_1$.

By saying X^\dagger (resp. $f : Y^\dagger \rightarrow X^\dagger$) has certain property, we mean an element (resp. a representative in the equivalence class) has such a property.

Proposition 3.3 (2.2.1, 2.2.2 [KL]). *Let $f : \bar{Y} \rightarrow \bar{X}$ be a finite flat morphism of rigid spaces with the ramification locus $Z \subset \bar{X}$. Let $X \subset \bar{X}$ be an admissible open quasi-compact subset such that $Z \cap X$ is a union of connected components of Z . If $Y \subset f^{-1}(X)$ is an admissible open such that $f|_Y$ is finite flat of degree d , then $(f|_Y)^\dagger : Y^\dagger \rightarrow X^\dagger$ is finite flat of degree d , which is moreover étale away from Z .*

As a consequence, if $f' : \bar{Y}' \rightarrow \bar{X}$ is another finite flat map, then any map $h : Y' := f'^{-1}(X) \rightarrow Y$ extends to $h^\dagger : Y'^\dagger \rightarrow Y^\dagger$, whose composition with $(f|_Y)^\dagger$ is the restrictions of f' .

3.1.3 Overconvergence

Let \bar{X} be a quasi-compact rigid space over K , whose formal model is denoted by $\tilde{\mathfrak{X}}$. Let $D \subset \tilde{\mathfrak{X}}_0$ be a Cartier divisor. Choose a finite covering $\{\mathfrak{U}_i\}_{i=1, \dots, n}$ of $\tilde{\mathfrak{X}}_0$ so that for any i the ideal of $D|_{\mathfrak{U}_i}$ is generated by a single section $h_i \in \mathcal{O}_{\tilde{\mathfrak{X}}_0}$. Choose for each h_i a lifting $\tilde{h}_i \in \Gamma(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}})$. For any $r \in (|p|^{1/e}, 1]$, define

$$\bar{X}(r) = \bigcup_{1 \leq i \leq n} \{x \in \bar{X} \mid |\tilde{h}_i| \geq r\}.$$

Proposition 3.4 (2.1.3, 2.3.1, 2.3.2 [KL]). *We have the following:*

- (1) *If $r \in (|p|^{1/e}, 1)$, then $\bar{X}(1) \Subset_{\bar{X}} \bar{X}(r)$.*
- (2) *Let $U \subset X \subset \bar{X}$ be admissible open quasi-compact subsets, and suppose $U \Subset_{\bar{X}} \bar{X}$. Then $U \Subset X$ if and only if $U \Subset_{\bar{X}} X$.*
- (3) *If $s, r \in (|p|^{1/e}, 1]$ with $s < r$, then $\bar{X}(r) \Subset_{\bar{X}} \bar{X}(s)$.*
- (4) *If $\bar{X}(1) \Subset_{\bar{X}} X'$ for X' a quasi-compact admissible open of \bar{X} , then for any r close enough to 1 we have $\bar{X}(r) \subset X'$.*

3.2 The canonical subgroup

Let K be a finite extension of \mathbf{Q}_p . There exists a functor $X \mapsto X^{\text{rig}}$ from the category of schemes which are locally of finite type over K to the category of K -rigid spaces, and a functor $\mathcal{F} \mapsto \mathcal{F}^{\text{rig}}$ sending a coherent sheaf on X to its rigidification on X^{rig} .

Let X be a scheme which is locally of finite type over \mathcal{O}_K , and \mathfrak{X} the formal completion of X along its special fiber. Denote by $\mathfrak{X}^{\text{rig}}$ the rigid fibre of \mathfrak{X} via Raynaud's functor. There is a natural morphism between the rigid fibre $\mathfrak{X}^{\text{rig}}$ and the analitification

$$X^{\text{an}} := (X \otimes_{\mathcal{O}_K} K)^{\text{an}}$$

of X :

$$\mathfrak{X}^{\text{rig}} \rightarrow X^{\text{an}},$$

which is an isomorphism if X is proper.

3.2.1 Passing to the rigid analytic picture

Write

$$\bar{\mathcal{M}}_{H^p} := \vec{\mathcal{M}}_{H^p}^{\text{ord,tor}}, \quad \mathcal{M}_{H^p}^* := \vec{\mathcal{M}}_{H^p}^{\text{ord,*}}$$

to simplify the notation.

Let $\mathfrak{M}_{H^p}^*$ (resp. $\bar{\mathfrak{M}}_{H^p}$) be the formal completion of $\mathcal{M}_{H^p}^*$ (resp. $\bar{\mathcal{M}}_{H^p}$) along the special fiber over p . By our notation, $\mathfrak{M}_{H^p}^{*,\text{rig}}$ and $\bar{\mathfrak{M}}_{H^p}^{\text{rig}}$ are the associated rigid spaces.

The same procedure of completion along the special fibre gives the universal semi-abelian schemes

$$\mathfrak{A}^* \rightarrow \mathfrak{M}_{H^p}^*, \quad \bar{\mathfrak{A}} \rightarrow \bar{\mathfrak{M}}_{H^p}.$$

Combining with rigidification, we get the universal semi-abelian scheme

$$\mathfrak{A}^{*,\text{rig}} \rightarrow \mathfrak{M}_{H^p}^{*,\text{rig}}, \quad \bar{\mathfrak{A}}^{\text{rig}} \rightarrow \bar{\mathfrak{M}}_{H^p}^{\text{rig}}.$$

Take \mathfrak{M}_{H^p} to be the open formal subscheme of $\mathfrak{M}_{H^p}^*$ whose points are in $\vec{\mathcal{M}}_{H^p}^{\text{ord}}$, which is hence equipped with the universal objects

$$\mathfrak{A} = \mathfrak{A}^*|_{\mathfrak{M}_{H^p}} \rightarrow \mathfrak{M}_{H^p}.$$

Set $\mathfrak{M}_{H^p}^{\text{rig}}$ to be the open subspace of $\bar{\mathfrak{M}}_{H^p}^{\text{rig}}$ lying over $\vec{\mathcal{M}}_{H^p}^{\text{ord,an}}$, which then comes with the restriction

$$\mathfrak{A}^{\text{rig}} = \bar{\mathfrak{A}}^{\text{rig}}|_{\mathfrak{M}_{H^p}^{\text{rig}}} \rightarrow \mathfrak{M}_{H^p}^{\text{rig}}.$$

3.2.2 Canonical subgroup and Frobenius

The complement D of the ordinary locus of the special fibre $(\bar{\mathcal{M}}_{H^p})_0$ of $\bar{\mathcal{M}}_{H^p}$ is a Cartier divisor, which is the vanishing locus of the Hasse invariant h . We then apply the construction in Section 3.1.3 to $\bar{X} = \bar{\mathcal{M}}_{H^p}^{\text{rig}}$. We then have for $r \in (|p|^{1/e}, 1]$ the quasi-compact admissible open

$$\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r) \subset \bar{\mathcal{M}}_{H^p}^{\text{rig}}.$$

In particular, we see by definition that

$$\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(1) = \mathfrak{M}_{H^p}^{\text{rig}}.$$

By Corollary 5.10(b) and Corollary 5.11 [Lü], we get easily, for $r, s \in (|p|^{1/e}, 1)$ with $s < r$,

$$\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r) \subseteq \bar{\mathfrak{M}}_{H^p}^{\text{rig}}(s).$$

and

$$\mathfrak{M}_{H^p}^{\text{rig}} \subseteq \bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r).$$

In particular, we have

$$\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r) \in \bar{\mathfrak{M}}_{H^p}^{\text{rig}, \dagger}.$$

One has the following result from [Fa] (see also 4.1.3 [AIP]).

Theorem 3.5 (Theorem 6, [Fa]). *For each $n \in \mathbf{Z}_{\geq 1}$ and r sufficiently close to 1, there is a canonical subgroup of level n $\mathcal{H}_n(r) \subset \bar{\mathfrak{A}}^{\text{rig}}[p^n]$ of the p -divisible group $G = \bar{\mathfrak{A}}^{\text{rig}}[p^\infty]$ over $\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r)$, which is locally free of rank p^{dg} , and whose restriction to the ordinary locus $\mathfrak{M}_{H^p}^{\text{rig}}$ is the multiplicative subgroup $\mathfrak{A}^{\text{rig}}[p^n]^\circ \subset \mathfrak{A}^{\text{rig}}[p^n]$.*

Moreover, the level-1 canonical subgroup $\mathcal{H}_1(r)$ is the kernel of the Frobenius on G , and for any $1 \leq m \leq n$, $\mathcal{H}_n(r)/\mathcal{H}_m(r)$ is the canonical subgroup of $G/\mathcal{H}_m(r)$ of level $n - m$.

The multiplicative subgroup $\mathfrak{A}[p^n]^\circ \subset \mathfrak{A}[p^n]$ is a finite flat group scheme of order p^{ndg} . One has, by the proof of 1.11.6 [Ka78], that $\mathfrak{A}/\mathfrak{A}[p^n]^\circ$ gives an element in \mathfrak{M}_{H^p} . We thus have a canonical map

$$\varphi^n : \mathfrak{M}_{H^p} \rightarrow \mathfrak{M}_{H^p}, \quad \mathfrak{A} \mapsto \mathfrak{A}/\mathfrak{A}[p^n]^\circ.$$

Proposition 3.6. *The morphism φ induces the following morphisms, which are all finite flat of degree p^{ndg} :*

- (1) $\varphi^n : \bar{\mathfrak{M}}_{H^p} \rightarrow \bar{\mathfrak{M}}_{H^p}$.
- (2) $\varphi^{n, \text{rig}} : \bar{\mathfrak{M}}_{H^p}^{\text{rig}} \rightarrow \bar{\mathfrak{M}}_{H^p}^{\text{rig}}$.
- (3) $\varphi^{n, \text{rig}, \dagger} : \bar{\mathfrak{M}}_{H^p}^{\text{rig}, \dagger} \rightarrow \bar{\mathfrak{M}}_{H^p}^{\text{rig}, \dagger}$.
- (4) $\varphi^n(r) : \bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r) \rightarrow \bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r^{p^n})$ which extends the map in (2), if r is close enough to 1.

Furthermore, the first two maps are étale over the open parts \mathfrak{M}_{H^p} and $\mathfrak{M}_{H^p}^{\text{rig}}$, respectively.

Proof. The map (1) is defined by $\bar{\mathfrak{A}} \mapsto \bar{\mathfrak{A}}/\bar{\mathfrak{A}}[p^n]^\circ$. Then the map (2) induced by the first one on the rigid fibre. The third map is by Theorem 3.5 for the level n canonical subgroup. The claim for map (4) follow from the argument in the proof of 3.1.7 [KL], together with the observation on abelian schemes in 1.11.4 [Ka78].

□

Corollary 3.7. *For r sufficiently close to 1, the sheaf $\bar{\mathfrak{A}}^{\text{rig}}[p^n]/\mathcal{H}_n(r)$ is finite flat.*

Proof. This is by Proposition 3.6 (4).

□

3.3 The formal Igusa tower

Following the idea of Hida (see e. g. [Hi02]), we define

$$\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}} = \text{Isom}_{\bar{\mathfrak{M}}_{H^p}}((\bar{\mathfrak{A}}[p^n]^\circ, \bar{\mathfrak{A}}^\vee[p^n]^\circ), ((D^0/p^n D^0)^{\text{mult}}, (D^{\vee,0}/p^n D^{\vee,0})^{\text{mult}}).$$

The formal scheme $\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}}$ is a Galois cover of $\bar{\mathfrak{M}}_{H^p}$ with Galois group $\text{GL}_g(\mathcal{O}/p^n \mathcal{O})$, the preimage of \mathfrak{M}_{H^p} under which is written as $\mathfrak{M}_{H^p p^n}^{\text{ur,ord}}$.

We then have a proper map

$$\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}} \rightarrow \bar{\mathfrak{M}}_{H^p} \rightarrow \mathfrak{M}_{H^p}^*,$$

for which the Stein factorization is written as $\mathfrak{M}_{H^p p^n}^{*,\text{ur,ord}}$. The rigid fibre of $\mathfrak{M}_{H^p p^n}^{*,\text{ur,ord}}$ is denoted by $\mathfrak{M}_{H^p p^n}^{*,\text{ur,ord,rig}}$, as usual.

In an analogous way, we have the universal semi-abelian scheme

$$\bar{\mathfrak{A}}_n \rightarrow \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}}$$

and the universal abelian scheme

$$\mathfrak{A}_n \rightarrow \mathfrak{M}_{H^p p^n}^{\text{ur,ord}}$$

by restriction.

Similarly as before, we have the associated rigid space $\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}}$ and the open subspace $\mathfrak{M}_{H^p p^n}^{\text{ur,ord,rig}}$ being the intersection of $\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}}$ with $\mathcal{M}_H^{\text{ord,an}} = \bar{\mathcal{M}}_H^{\text{ord,an}}$, where for H , H_p has been chosen to provide the level structure at p of $\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}}$. We then have the finite étale map

$$\text{Ig}_n : \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}} \rightarrow \bar{\mathfrak{M}}_{H^p}^{\text{rig}},$$

which is a Galois cover of $\bar{\mathfrak{M}}_{H^p}^{\text{rig}}$.

For r close enough to 1, we set

$$\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r)$$

to be the Galois cover of $\bar{\mathfrak{M}}_{H^p}^{\text{rig}}(r)$ trivializing the finite flat sheaf $\bar{\mathfrak{A}}^{\text{rig}}[p^n]/\mathcal{H}_n(r)$ (see Corollary 3.7) and its dual. In particular, the ordinary locus $\mathfrak{M}_{H^p p^n}^{\text{ur,ord,rig}} := \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(1)$ is the rigid fibre of the formal Igusa tower $\mathfrak{M}_{H^p p^n}^{\text{ur,ord}}$ constructed above, which justifies the notation. We set

$$\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(r) = \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r) \cap (\mathcal{M}_H^{\text{ord}})^{\text{an}},$$

for r close enough to 1.

For $s < r$ with s close enough to 1, we have

$$\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r) \subseteq \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(s), \tag{3.3.1}$$

by Proposition 3.2 and the known fact in the case $n = 0$.

Again by the proof of 1.11.6 [Ka78], we have a well-defined map

$$\varphi_n : \mathfrak{M}_{H^p p^n}^{\text{ur,ord}} \rightarrow \mathfrak{M}_{H^p p^n}^{\text{ur,ord}}, \quad \mathfrak{A}_n \mapsto \mathfrak{A}_n / \mathfrak{A}_n[p]^\circ.$$

sitting in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_{H^p p^n}^{\text{ur,ord}} & \xrightarrow{\varphi_n} & \mathfrak{M}_{H^p p^n}^{\text{ur,ord}} \\ \downarrow \text{Ig}_n & & \downarrow \text{Ig}_n \\ \mathfrak{M}_{H^p} & \xrightarrow{\varphi=\varphi^1} & \mathfrak{M}_{H^p} \end{array}$$

Proposition 3.8. *The map φ_n induces, for r close enough to 1, the following morphisms which are all finite flat of degree p^{dg} :*

- (1) $\bar{\varphi}_n : \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}} \rightarrow \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}}.$
- (2) $\bar{\varphi}_n^{\text{ur,ord,rig}} : \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}} \rightarrow \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}}.$
- (3) $\bar{\varphi}_n^{\text{ur,rig},\dagger} : \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig},\dagger} \rightarrow \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig},\dagger}.$
- (4) $\bar{\varphi}_n(r) : \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r) \rightarrow \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r^p).$

Proof. (1) and (2) are proved in the same way as Proposition 3.6 (1) and (2), and (3) is by the finite flatness of Ig_n and Proposition 3.3. Finally (4) is proved by the use of Proposition 3.3 and the argument in the proof of 3.1.7 [KL], for which one is referred to the proof of 3.2.6 [KL] for details. \square

4 Overconvergence of Siegel-Hilbert modular forms

4.1 p -adic Banach spaces

Set $\bar{\mathcal{Z}}^{\text{ord,rig}} \subset \bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}}$ to be the tube of $(\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}})_0 \setminus (\mathfrak{M}_{H^p p^n}^{\text{ur,ord}})_0$, and set $\mathcal{Z}^{\text{ord,rig}}$ to be its intersection with $\mathfrak{M}_{H^p p^n}^{\text{ur,ord,rig}}$.

Lemma 4.1 (Köcher principle). *For $r < 1$ which is sufficiently close to 1, we have the natural isomorphisms*

$$H^0(\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r), (\det \omega)^\kappa) \xrightarrow{\sim} H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(r), (\det \omega)^\kappa) \xrightarrow{\sim} H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(r) \setminus \mathcal{Z}^{\text{ord,rig}}, (\det \omega)^\kappa).$$

Proof. As is explained in the proof of 4.1.4 [KL], it suffices to show the natural map

$$H^0(\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord}}, (\det \omega)^\kappa) \rightarrow H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,ord}}, (\det \omega)^\kappa)$$

is an isomorphism. By the existence of the Galois covering of $\mathfrak{M}_H^{\text{ord}}$ by $\mathfrak{M}_{H^p p^n}^{\text{ur,ord}}$, we just need to show the isomorphism for $n = 0$, which in turn follows from Proposition 4.9 [Ra78].

□

Recall we have the connected component $\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)$ of $\mathfrak{M}_{H^p p^n}^{v, \text{ur}, \text{rig}}(r)$ for any prime ideal $v \subset \mathcal{O}$ and double coset γ_v . Define $\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)$ by the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r) & \longrightarrow & \mathcal{M}_H^{\gamma_v, \text{ord}, \text{an}} \\ \downarrow & & \downarrow \pi_{1, \gamma_v} \\ \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r) & \longrightarrow & \mathcal{M}_H^{\text{ord}, \text{an}} \end{array}$$

Note that the left vertical map is finite étale, being the base change of the finite étale map π_{1, γ_v} . We also denote it by π_{1, γ_v} .

Proposition 4.2. *Let $r < 1$ be sufficiently close to 1. We have that $\mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r))$ is a p -adic Banach space with respect to the norm*

$$|f|_r := \sup_{x \in \mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)} |f(x)|$$

for a function $f \in \mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r))$. This is the same as the norm

$$|f|_r^\circ := \sup_{x \in \mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r) \setminus \pi_{1, \gamma_v}^{-1}(\mathcal{Z}^{\text{ord}, \text{rig}})} |f(x)|.$$

Proof. We use the argument in the proof of 4.1.6 [KL].

First note that $|f|_r$ is a well-defined norm on the p -adic Banach space $\mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r) \setminus \pi_{1, \gamma_v}^{-1}(\mathcal{Z}^{\text{ord}, \text{rig}}))$, with the latter space being finite over $\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur}, \text{rig}}(r) \setminus \bar{\mathcal{Z}}^{\text{ord}, \text{rig}}$ and hence quasi-compact. (If \mathcal{F} is a coherent sheaf on a quasi-compact rigid space X , then $\mathcal{F}(X)$ is a p -adic Banach space.)

Then we only need to show that

$$|f|_r = |f|_r^\circ, \quad \forall f \in \mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)), \quad (4.1.1)$$

which will then realize

$$\mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)) \hookrightarrow \mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r) \setminus \pi_{1, \gamma_v}^{-1}(\mathcal{Z}^{\text{ord}, \text{rig}}))$$

as a closed subspace. For this, recall we have the Köcher principle and the fact that the natural projection

$$\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{ord}, \text{rig}} \xrightarrow{\pi_{1, \gamma_v}} \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{ord}, \text{rig}} \xrightarrow{\text{Ig}_n} \mathfrak{M}_{H^p}^{\text{rig}}$$

is finite étale. The former is Lemma 4.1, and the latter holds because both π_{1, γ_v} and Ig_n are finite étale by construction. Then by the argument in the proof of 4.1.6 [KL], we are reduced to show (4.1.1) for $r = 1$ and $f \in \mathcal{O}(\bar{\mathfrak{M}}_{H^p})$. In this case we may assume that its image $f_0 \in \mathcal{O}(\bar{\mathfrak{M}}_{H^p})_0$ is non-zero. If $|f|_1^\circ < |f|_1 = 1$, then f_0 is nilpotent and then vanishes on the open subset $(\mathfrak{M}_{H^p})_0$ of $(\bar{\mathfrak{M}}_{H^p})_0$. This, however, implies that f_0 vanishes on the whole $(\bar{\mathfrak{M}}_{H^p})_0$ as the latter is normal, a contradiction.

□

Corollary 4.3. *Let $\theta : \pi_{2,\gamma_v}^*((\det \omega)^{k_0(p-1)}) \rightarrow \pi_{1,\gamma_v}^*((\det \omega)^{k_0(p-1)})$ be the canonical map of sheaves on $\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{ord}, \text{rig}}$. Then for r sufficiently close to 1,*

$$\theta(\pi_{2,\gamma_v}^*(E)) = f_{k_0} \pi_{1,\gamma_v}^*(E) \in H^0(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{ord}, \text{rig}}(r), \pi_{1,\gamma_v}^*((\det \omega)^{k_0(p-1)}))$$

for some $f_{k_0} \in \mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{ord}, \text{rig}}(r))$.

Moreover, we have that $f_{k_0} - 1$ is topologically nilpotent. In fact, for any $0 < \epsilon \leq 1$, there is an $r \leq 1$ such that $|f_{k_0} - 1|_r \leq |p|^\epsilon$.

Proof. One sees that the proof of 4.1.7 of [KL] applies here. We only give a sketch here. The first assertion follows from the fact that the special fibre $(\mathfrak{M}_{H^p}^*)_0$ is normal. To show the second assertion, first note that the case $r = 1$ follows from Lemma 2.10 and Proposition 4.2. The case for a general r then follows from the $r = 1$ case, as well as Proposition 3.4(4). \square

4.2 Hecke correspondences on the strict neighbourhoods

Using Corollary 4.3 we now show the following:

Proposition 4.4. (1) *We have the inclusion*

$$\pi_{2,\gamma_v}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{ord}, \text{rig}}) \subset \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{ord}, \text{rig}},$$

and the inclusion

$$\pi_{2,\gamma_v}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)) \subset \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r)$$

for $r \in (|p|, 1)$.

In particular, the inclusions above hold for $\pi_{2,\mathfrak{a}}$ on $\mathfrak{M}_{H^p p^n}^{\mathfrak{a}, \text{ur}, \text{ord}, \text{rig}}$, with $\mathfrak{a} \subset \mathcal{O}$ an ideal.

(2) For $r \rightarrow 1^-$, $\pi_{2,(p)}$ induces a map

$$\mathfrak{M}_{H^p p^n}^{(p), \text{ur}, \text{rig}}(r^p) \rightarrow \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r).$$

Proof. (1) The first assertion follows from the construction of $\mathfrak{M}_{H^p p^n}^{\text{ur}, \text{ord}, \text{rig}}$.

For the second inclusion, we argue as in the proof of 4.1.10 [KL]. First observe that it is enough to show this for E -valued points for any finite extension E/\mathbf{Q}_p . Let $f : A \rightarrow B$ be an element in $\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur}, \text{rig}}(r)(E)$. We may enlarge E so that A extends to a semi-abelian scheme \bar{A} over \mathcal{O}_E . We may and do assume that \bar{A} is an abelian scheme, because otherwise $r = 1$ and we go back to the first assertion. Then we can extend $\text{Ker}(f)$ to a finite flat subgroup scheme of \bar{A} . The quotient of \bar{A} by this subgroup scheme is denoted by \bar{B} , and the projection $\bar{A} \rightarrow \bar{B}$ by pr . Let $\mathbf{v}_{\bar{A}}$ (resp. $\mathbf{v}_{\bar{B}}$) be a basis of $H^0(\bar{A}, \Omega_{\bar{A}/\mathcal{O}_E}^1)$ (resp. $H^0(\bar{B}, \Omega_{\bar{B}/\mathcal{O}_E}^1)$). Then we must have

$$pr^*(\det \mathbf{v}_{\bar{B}}) = a \det \mathbf{v}_{\bar{A}}$$

for some $a \in \mathcal{O}_E$.

Now by the definition of overconvergence, it suffices to show

$$|E(A, \det \mathbf{v}_{\bar{A}})| \leq |E(B, \det \mathbf{v}_{\bar{B}})|.$$

By Corollary 4.3, we have

$$E(B, \det \mathbf{v}_{\bar{B}}) = f_{k_0} E(A, pr^*(\det \mathbf{v}_{\bar{B}})),$$

with $f_{k_0} - 1$ topologically nilpotent. On the other hand, we have the equality

$$E(A, pr^*(\det \mathbf{v}_{\bar{B}})) = a^{(1-p)k_0} E(A, \det \mathbf{v}_{\bar{A}}).$$

Now the result follows because $|f_{k_0}(A, \mathbf{v}_{\bar{A}})| = 1$ and $|a^{(1-p)k_0}| \geq 1$.

(2) This is proved as in 4.3.3 [KL]. By Proposition 3.8, we have for s close enough to 1 the natural map

$$\bar{\varphi}_n(s) : \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s) \rightarrow \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s^p),$$

by restricting to the intersection of the map in Proposition 3.8 (4) with $\mathcal{M}_H^{\text{ord,an}}$. For $r \rightarrow 1^-$, by part (1) we see $\pi_{2,(p)}$ induces a map

$$\mathfrak{M}_{H^p p^n}^{(p),\text{ur,rig}}(r^p) \rightarrow \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s)$$

for some $r^p \leq s$. If $s \geq r$, we are done. Now assume $s < r$. Thus it suffices to show that their composition

$$\mathfrak{M}_{H^p p^n}^{(p),\text{ur,rig}}(r^p) \xrightarrow{\pi_{2,(p)}} \mathfrak{M}_{H^p p^n}^{\text{ur,ord,rig}}(s) \xrightarrow{\bar{\varphi}_n(s)} \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s^p)$$

factors through $\mathfrak{M}_{H^p p^n}^{\text{ur,ord,rig}}(r^p) \subset \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s^p)$, for which it is in turn enough to show the further composition with $\mathfrak{M}_{H^p}^{\text{rig}}(s^p)$

$$\mathfrak{M}_{H^p p^n}^{(p),\text{ur,rig}}(r^p) \xrightarrow{\pi_{2,(p)}} \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s) \xrightarrow{\bar{\varphi}_n(s)} \mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s^p) \xrightarrow{\text{Ig}_n} \mathfrak{M}_{H^p}^{\text{rig}}(s^p)$$

factors through $\mathfrak{M}_{H^p}^{\text{rig}}(r^p)$.

Tracing the construction we know that for $r \rightarrow 1^-$ the latter composition is induced by the map

$$\mathfrak{M}_{H^p p^n}^{(p),\text{ur,ord}} \xrightarrow{\pi_{1,(p)}} \mathfrak{M}_{H^p p^n}^{\text{ur,ord}} \xrightarrow{\text{Ig}_n} \mathfrak{M}_{H^p} \xrightarrow{[\cdot p]} \mathfrak{M}_{H^p},$$

where the map $[\cdot p]$ is the multiplication by p on the level structure of the universal semi-abelian scheme. To see this, using Proposition 3.4 (4), we only have to check the statement for the case $r = 1$, which is easily seen. Meanwhile, we notice that $[\cdot p]$ induces the identity map on Hasse invariant, because the Hasse invariant is independent of level structures. Hence it maps $\mathfrak{M}_{H^p}^{\text{rig}}(r^p)$ to itself. This concludes the proof. \square

4.3 Hecke operators on overconvergent Siegel-Hilbert modular forms

Let L/\mathbf{Q}_p be a finite extension and \mathcal{R} an L -affinoid algebra with a fixed sub-multiplicative semi-norm extending the norm on L , and $Y \in \mathcal{R}$ such that

$$|Y| < |p|^{\frac{1}{p-1}-1}.$$

We define the spaces of overconvergent Siegel-Hilbert modular forms of level $H^p p^n$ and of weight κ (see Section 2.5.2) (reps. $\kappa + Y$) with coefficients in L (reps. in \mathcal{R}) as follows:

$$\begin{aligned} M_{H^p p^n, \kappa}^\dagger(L) &= \varinjlim_{r \rightarrow 1^{-1}} M_{H^p p^n, \kappa, r}(L), \\ M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R}) &= \varinjlim_{r \rightarrow 1^{-1}} M_{H^p p^n, \kappa+Y, r}(\mathcal{R}), \end{aligned}$$

where we use the convention that for a \mathbf{Q}_p -analytic space X ,

$$X_{\mathcal{R}} = X \times_{\mathrm{Sp} \mathbf{Q}_p} \mathrm{Sp} \mathcal{R},$$

and

$$M_{H^p p^n, \kappa+Y, r}(\mathcal{R}) = H^0(\mathfrak{M}_{H^p p^n}^{\mathrm{ur}, \mathrm{rig}}(r)_{\mathcal{R}}, \omega^\kappa).$$

By Proposition 4.4 (1), we have two projections

$$\pi_{1, \gamma_v}, \pi_{2, \gamma_v} : \mathfrak{M}_{H^p p^n}^{\gamma_v, \mathrm{ur}, \mathrm{rig}}(r) \longrightarrow \mathfrak{M}_{H^p p^n}^{\mathrm{ur}, \mathrm{rig}}(r).$$

We then get the (pull-back of) canonical map of sheaves on $\mathfrak{M}_{H^p p^n}^{\gamma_v, \mathrm{ur}, \mathrm{rig}}(r)_{\mathcal{R}}$

$$\pi_{2, \gamma_v}^* \omega^\kappa \longrightarrow \pi_{1, \gamma_v}^* \omega^\kappa,$$

whose composition with multiplication by $f_{k_0}^{\frac{Y}{k_0(p-1)}}$ gives the map

$$\pi_{2, \gamma_v}^* \omega^\kappa \longrightarrow \pi_{1, \gamma_v}^* \omega^\kappa \xrightarrow{\cdot f_{k_0}^{\frac{Y}{k_0(p-1)}}} \pi_{1, \gamma_v}^* \omega^\kappa.$$

Here

$$f_{k_0}^{\frac{Y}{k_0(p-1)}} := \exp\left(\frac{Y}{k_0(p-1)} \log f_{k_0}\right)$$

is a well-defined element in $\mathcal{O}(\mathfrak{M}_{H^p p^n}^{\gamma_v, \mathrm{ur}, \mathrm{rig}}(r)_{\mathcal{R}})$ for r such that $|f_{k_0} - 1|_r \leq |p|^\epsilon$, where ϵ is chosen to satisfy $|Y||p|^\epsilon < |p|^{1/(p-1)}$. We remark that the analyticity of $f_{k_0}^{\frac{Y}{k_0(p-1)}}$ follows from the assumption that $|Y| < |p|^{\frac{1}{p-1}-1}$ (and the assumption that $p \nmid k_0$).

Applying π_{1, γ_v}^* to the above map, and composing it with the trace map, we obtain

$$\pi_{1, \gamma_v}^* \pi_{2, \gamma_v}^* \omega^\kappa \longrightarrow \pi_{1, \gamma_v}^* \pi_{1, \gamma_v}^* \omega^\kappa \xrightarrow{\pi_{1, \gamma_v}^* \cdot f_{k_0}^{\frac{Y}{k_0(p-1)}}} \pi_{1, \gamma_v}^* \pi_{1, \gamma_v}^* \omega^\kappa \xrightarrow{\mathrm{tr}} \omega^\kappa.$$

Then taking global sections, we get an endomorphism T_{γ_v} of $H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(r)_{\mathcal{R}}, \omega^\kappa)$. The resulting Hecke operator will be denoted by $T_{v,i}$ (resp. S_v) if $\gamma_v = T_{v,i}$ (resp. S_v).

Letting $r \rightarrow 1^-$, we obtain the Hecke operator on $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$, which is denoted by the same symbol. The ring of endomorphisms generated by all the Hecke operators (with v and γ_v varying) is denoted by $\mathbf{T}_{H^p p^n, \kappa+Y}^\dagger$. The product of $T_{v,1}$'s for all $v|p$ is denoted by $U_{(p)}$.

The construction made above and Corollary 4.3 thus give rise to the following

Proposition 4.5. *Let $\psi_t : \mathcal{R} \rightarrow L'$ be the character to a finite extension L'/L , which sends Y to $(p-1)k_0 t$ for some $t \in \mathbf{Z}_{\geq 0}$. We have the following commutative diagram compatible with the actions of Hecke operators T :*

$$\begin{array}{ccccc} M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R}) & \xrightarrow{\text{Id} \otimes \psi_t} & M_{H^p p^n, \kappa}^\dagger(L') & \xrightarrow{\cdot E^t} & M_{H^p p^n, \kappa \cdot \text{Nm}(p-1)k_0 t}^\dagger(L') \\ T \downarrow & & & & T \downarrow \\ M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R}) & \xrightarrow{\text{Id} \otimes \psi_t} & M_{H^p p^n, \kappa}^\dagger(L') & \xrightarrow{\cdot E^t} & M_{H^p p^n, \kappa \cdot \text{Nm}(p-1)k_0 t}^\dagger(L') \end{array} \quad (4.3.1)$$

Proof. Let $T = T_{\gamma_v}$. Then we are supposed to check that for $f \otimes x \in M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$,

$$\psi_t(T(f \otimes x)) \cup E^t = T(f \otimes \psi_t(x) \cup E^t). \quad (4.3.2)$$

(This suffices for the proof since the T_{γ_v} 's generate the Hecke algebra.)

Now unwinding the definition of the Hecke operator T , we see, for $A \rightarrow B$ an isogeny in $\mathfrak{M}_{H^p p^n}^{\gamma_v, \text{ur,rig}}(r)$,

$$T(f \otimes x)(A) = f(B) \otimes f_{k_0}^{\frac{Y}{k_0(p-1)}} x.$$

Then, applied to $(A \rightarrow B)$, the left hand side of the equality (4.3.2) is equal to

$$f(B) \otimes \psi_t(x) \cup E^t(B),$$

while the right hand side is equal to

$$(f(B) \otimes \psi_t(x)) \cup f_{k_0}^t E^t(A),$$

since $\psi_t(Y) = k_0(p-1)t$. Now the result follows from Corollary 4.3. □

Remark 4.6. *In the notation of Proposition 4.5, the points in the image of the composition $E^t \cdot \text{Id} \otimes \psi_t$ are classical Siegel-Hilbert modular forms.*

Proposition 4.7. *For $s < r < 1$ with s sufficiently close to 1, the following natural inclusion is completely continuous:*

$$\text{Res}(s, r) : H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(s), \omega^\kappa) \hookrightarrow H^0(\mathfrak{M}_{H^p p^n}^{\text{ur,rig}}(r), \omega^\kappa).$$

Proof. It is equivalent to showing this for the natural inclusion

$$H^0(\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,ord,rig}}(s), \omega^\kappa) \hookrightarrow H^0(\bar{\mathfrak{M}}_{H^p p^n}^{\text{ur,rig}}(r), \omega^\kappa)$$

by the K ocher principle Lemma 4.1.

Recall $\mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r) \subseteq \mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(s)$ from (3.3.1). Now one concludes by Proposition 2.4.1 [KL]. \square

Lemma 4.8. *Suppose r is close enough to 1. The Hecke operator $U_{(p)}$ on the \mathcal{R} -module $M_{H^p p^n, \kappa+Y, r}(\mathcal{R})$ can be constructed as the composition of the natural inclusion $\text{Res}(r^p, r)$ and the following map induced by $\pi_{2, (p)}$:*

$$H^0(\mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r), \omega^\kappa) \rightarrow H^0(\mathfrak{M}_{H^p p^n}^{\text{ur}, \text{rig}}(r^p), \omega^\kappa). \quad (4.3.3)$$

Proof. This follows from Proposition 4.4 (2). \square

Corollary 4.9. *For r close enough to 1, the action of $U_{(p)}$ on $M_{H^p p^n, \kappa+Y, r}(\mathcal{R})$ is completely continuous.*

Proof. We know by Proposition 4.7 that the map $\text{Res}(r, r^p)$ is completely continuous. Moreover, the map (4.3.3) is continuous. Since a composition of a continuous map followed by a completely continuous one is again completely continuous, we are done. \square

Remark 4.10. *It is the eigenvalues of the Hecke operator $U_{(p)}$ that we will interpolate, since we only construct a one parameter family of overconvergent Siegel-Hilbert eigenforms.*

4.4 Constructing families of overconvergent Siegel-Hilbert modular forms

4.4.1 The set up

Recall that T_g/\mathbf{Z} is the standard diagonal maximal torus of $\text{GSp}_{2g}/\mathbf{Z}$. Denote by $c : T_g \rightarrow \mathbf{G}_m$ the character:

$$c : \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_g & & & \\ & & & ba_1^{-1} & & \\ & & & & \ddots & \\ & & & & & ba_g^{-1} \end{pmatrix} \mapsto a_1 \cdots a_g b^{-2}.$$

Let \mathcal{W} be the rigid space whose E -valued points are continuous homomorphisms in

$$\text{Hom}_{\text{cont}}(T_g(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p), E^\times)$$

for any (not necessarily finite) field extension E/\mathbf{Q}_p .

In the rest of the paper, we fix a classical weight κ and a finite extension L/\mathbf{Q}_p . For our purpose we only need the part of the weight space that “differs” from our fixed weight κ only by parallel weights. Thus let \mathcal{W}_κ be the admissible subspace of \mathcal{W} whose E -valued points, for $E \subset \mathbf{C}_p$ a closed subfield containing L , are

$$\mathcal{W}_\kappa(E) = \{\chi = \kappa \cdot (\tau \circ c \circ \text{Nm}) : T_g(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p) \rightarrow E^\times\}$$

for some continuous character $\tau : \mathbf{Z}_p^\times \rightarrow E^\times$ satisfying

$$v_p(1 - \tau(t)) > \frac{1}{p-1}, \quad t \in \mathbf{Z}_p^\times.$$

Let

$$\chi^{\text{univ}} : T_g(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p) \rightarrow \mathcal{O}(\mathcal{W}_\kappa)^\times$$

be the universal such character.

We define a rigid analytic function Y on \mathcal{W}_κ as follows: if $\chi \in \mathcal{W}_\kappa(E)$ is as above, and is associated to $\tau : \mathbf{Z}_p^\times \rightarrow E^\times$, then the value of Y at χ is given by

$$Y(\chi) = \frac{\log \tau(t)}{\log t}$$

for $t \in \mathbf{Z}_p^\times$ sufficiently close to the identity.

By the construction above, we have that $|Y| < |p|^{\frac{1}{p-1}-1}$, hence the Banach module $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$ of overconvergent forms is well-defined, for any $\text{Sp } \mathcal{R} \subset \mathcal{W}_\kappa$. Let $\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$ be the closure of the ring of endomorphisms on $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$ generated by the Hecke operators at the places away from the level, under the norm defined in Proposition 4.2.

Proposition 4.11. *The \mathcal{R} -algebra $\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$ is commutative.*

Proof. This is proved using Proposition 4.5 and the ideas in the proof of 4.4.2 [KL] and 3.0.6.2 [Ti06]. Let $\mathcal{W}_\kappa^{\text{cl}}$ be a Zariski dense set of integral weights in the Y -neighbourhood of κ , which can be achieved by taking the parameters t to be sufficiently large powers of p , by Proposition 4.5. By the analyticity obtained in Proposition 4.5, each element in $\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$ is determined by the Zariski dense set $\mathcal{W}_\kappa^{\text{cl}}$. We then have the injection

$$\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}} \hookrightarrow \prod_{w \in \mathcal{W}_\kappa^{\text{cl}}} \mathbf{T}_{H^p p^n, w}^{\dagger, \text{ur}},$$

where each factor is the specialization. On the other hand, each Hecke ring $\mathbf{T}_{H, p^n, w}^{\dagger, \text{ur}}$ with the fixed integral weight w is commutative, being the completion of a commutative algebra of Hecke correspondences. Thus the product over $\mathcal{W}_\kappa^{\text{cl}}$ is commutative, hence so is its subring $\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$. □

Let

$$Z_\kappa = \text{Sp } \mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$$

be the rigid space over L associated to the \mathcal{R} -algebra $\mathbf{T}_{H^p p^n, \kappa+Y}^{\dagger, \text{ur}}$. It comes with the weight map

$$\underline{w} : Z_\kappa \rightarrow \text{Sp } \mathcal{R} \subset \mathcal{W}_\kappa.$$

Define

$$X_\kappa = Z_\kappa \times \mathbb{G}_m.$$

Write x_p for the canonical co-ordinate on \mathbb{G}_m .

4.4.2 Construction by the Coleman-Mazur machinery

Now we are ready to define the one parameter families of overconvergent Siegel-Hilbert modular form of level $H^p p^n$ as Coleman and Mazur proceed in [CM98]. For our purpose it is enough to to construct it over any affinoid quasi-compact subset $\mathrm{Sp}\mathcal{R} \subset \mathcal{W}_\kappa$. We fix such an \mathcal{R} from now on.

Set \mathcal{H} to be the (topological) commutative ring generated by the formal variables $X_{(p)}$, together with $X_{v,i}, Y_v$ (here $i = 1, \dots, g$) for all prime ideals $v \subset \mathcal{O}$ away from the level. Let

$$\iota : \mathcal{H} \rightarrow \mathcal{O}(X_\kappa)$$

be the map sending:

$$X_{v,i} \mapsto t_{v,i},$$

$$Y_v \mapsto s_v,$$

$$X_{(p)} \mapsto x_p.$$

Here we have denoted by $t_{v,i}$ and s_v the image of the Hecke operators $T_{v,i}$ and S_v in $\mathcal{O}(X_\kappa)$ respectively, regarded as functions on Z_κ . Then \mathcal{H} acts on $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$ with the action factoring through $\iota(\mathcal{H})$.

For r sufficiently close to 1, we know by Corollary 4.9 that $U_{(p)}$ acts on $(M_{H^p p^n, \kappa+Y, r})(\mathcal{R})$ completely continuously. This implies that the action of $\iota(\alpha)U_{(p)}$ on $M_{H^p p^n, \kappa+Y, r}(\mathcal{R})$ is completely continuous for any $\alpha \in \mathcal{H}$. Following Section 4 of [CM98], we can, for each $\alpha \in \mathcal{H}$, form the Fredholm series

$$P_\alpha(T) = \det_{\mathcal{R}}(1 - \iota(\alpha)U_{(p)}T | M_{H^p p^n, \kappa+Y, r}(\mathcal{R})) \in \mathcal{R}[[T]],$$

which is independent of the choice of r (for r sufficiently close to 1) by the lemma below.

Lemma 4.12. *Let $0 < r < r' < 1$ with r sufficiently close to 1. Then the Banach \mathcal{R} -module $M_{H^p p^n, \kappa+Y, r}(\mathcal{R})$ admits an orthogonal basis, which is also an orthogonal basis for the \mathcal{R} -submodule $M_{H^p p^n, \kappa+Y, r'}(\mathcal{R})$.*

Proof. Since $M_{H^p p^n, \kappa+Y, r}(\mathcal{R}) = M_{H^p p^n, \kappa+Y, r}(\mathbf{Q}_p) \hat{\otimes}_{\mathbf{Q}_p} \mathcal{R}$, we may assume $\mathcal{R} = \mathbf{Q}_p$ and $Y = 0$. Recall that, as r is sufficiently close to 1, the natural map $\mathfrak{M}_{H^p p^n}^{*, \mathrm{ur}, \mathrm{rig}}(r) \rightarrow \mathfrak{M}_{H^p}^{*, \mathrm{rig}}(r)$ is finite étale. We can conclude the proof by 2.4.5 [KL], namely in the notation of *loc. cit.* we let \mathcal{F} be the push-forward of $(\det \omega)^\kappa$ under the composition (recall the notation from Section 3.3)

$$\bar{\mathfrak{M}}_{H^p p^n}^{\mathrm{ur}, \mathrm{rig}}(r) \rightarrow \mathfrak{M}_{H^p p^n}^{*, \mathrm{ur}, \mathrm{rig}}(r) \rightarrow \mathfrak{M}_{H^p}^{*, \mathrm{rig}}(r).$$

Moreover, we have the line bundle $\mathcal{L} = (\det \omega)^{p-1}$ which is ample over $\mathcal{M}_{H^p}^*$, and $\mathcal{D} \subset (\mathcal{M}_{H^p}^*)_{\mathbf{F}_p}$ the divisor where the Hasse invariant h vanishes. Then 2.4.5 [KL] gives the result we require. \square

Set

$$\mathcal{E}_\kappa = \mathcal{E}_\kappa^{\mathrm{red}} \subset X_\kappa$$

to be the nilreduction of the Zariski-closed subspace of X_κ cut out by the ideal generated by the functions $P_\alpha((x_p \iota(\alpha))^{-1})$ for all the $\alpha \in \mathcal{H}$ such that $\iota(\alpha)$ is a unit. Alternatively, we can define \mathcal{E}_κ as follows.

The entire series associated to $\alpha \in \mathcal{H}$, $P_\alpha(T) \in \mathcal{R}[[T]]$, defines a closed subspace

$$\mathcal{Z}_\alpha \subset \mathrm{Sp} \mathcal{R} \times \mathbf{A}^1,$$

where T is regarded as the co-ordinate on \mathbf{A}^1 . For each $\alpha \in \mathcal{H}$ such that $\iota(\alpha)$ is unit. We can define the map

$$r_\alpha : X_\kappa = Z_\kappa \times \mathbf{G}_m \rightarrow \mathrm{Sp} \mathcal{R} \times \mathbf{A}^1 : x = (z, s) \mapsto (\underline{w}(z), \frac{s}{(\iota(\alpha))(x)}),$$

where we have regarded $\iota(\alpha)$ as a function on X_κ . Then we set \mathcal{E}_κ to be the nilreduction of

$$\bigcap_{\substack{\alpha \in \mathcal{H}, \\ \iota(\alpha) \in \mathcal{O}(X_\kappa)^\times}} r_\alpha^{-1}(\mathcal{Z}_\alpha).$$

The following theorem is obtained from the above construction formally, as in [CM98].

Theorem 4.13. (1) Let $E \subset \mathbf{C}_p$ be a closed subfield containing L . For an E -valued point $x \in \mathcal{E}_\kappa(E)$, there is a non-zero simultaneous eigenvector $f_x \in M_{H^p p^n, \kappa + Y(x)}^\dagger(E)$ for all the Hecke operators in $\mathbf{T}_{H^p p^n, \kappa + Y}^\dagger$ such that the Hecke eigenvalues $\lambda_{T_{v,i}}(z)$, $\lambda_{S_v}(x)$, $\lambda_{U_{(p)}}(x)$ for the operators $T_{v,i}$, S_v and $U_{(p)}$ satisfy:

$$\lambda_{T_{v,i}}(x) = t_{v,i}(x),$$

$$\lambda_{S_v}(x) = s_v(x),$$

$$\lambda_{U_{(p)}}(x) = x_p(x).$$

For a fixed $Y_0 \in E$ with $v_p(Y_0 - 1) > \frac{1}{p-1}$, the above assignment induces a bijection between the points $\{x \in \mathcal{E}_\kappa(E)\}_{Y(x)=Y_0}$ and systems of $\mathbf{T}_{H^p p^n, \kappa + Y}^\dagger$ -eigenvalues of an eigenvector $f \in M_{H^p p^n, \kappa + Y_0}^\dagger(E)$ of finite slope at p .

(2) The rigid analytic space \mathcal{E}_κ is a curve. The weight map $\underline{w} : \mathcal{E}_\kappa \rightarrow \mathcal{W}_\kappa$ is, locally in the domain, finite flat. The image of any component of \mathcal{E}_κ under this map misses at most finitely many points in \mathcal{W}_κ .

The following theorem plays a similar (yet weaker) role as the expected result that classical Siegel-Hilbert eigenforms are Zariski dense in the rigid analytic space \mathcal{E}_κ .

Theorem 4.14. Let f be a classical Siegel-Hilbert modular eigenform of weight κ and of level $H^p p^n$. There exists, for any positive integer t with $v_p(t)$ large enough, a Siegel-Hilbert modular eigenform f_t of weight $\kappa \cdot \mathrm{Nm}^{(p-1)k_0 t}$ and of the same level, such that the Hecke eigenvalues on the f_t 's converge p -adically to that of f , as $v_p(t) \rightarrow +\infty$. Furthermore, if f is cuspidal, then the f_t can also be taken to be cuspidal.

Proof. The proof is completely similar to that of 4.5.6 [KL].

As before, we take the weight space \mathcal{W}_κ centered in κ to be $\mathrm{Sp} \mathcal{R}$ since the construction is local. By the construction of \mathcal{E}_κ , when $\iota(\alpha)$ is unit, we have a map

$$r_\alpha : \mathcal{E}_\kappa \longrightarrow \mathcal{Z}_\alpha.$$

By the method of Chapter 7 [CM98], we see the projection r_α is finite.

Let $x \in \mathcal{E}_\kappa(L)$ be the point corresponding to f . By the argument of 6.2.2 and 6.3.2 [CM98] we may assume

$$r_\alpha^{-1}(r_\alpha(x)) = \{x\}. \quad (4.4.1)$$

By this property, we only need to find a family of elements in $\mathcal{Z}_\alpha(L)$ converging to $r_\alpha(x) := x_0$.

Let $w \in \mathrm{Sp} \mathcal{R}$ denote the weight of x . Let x_1, \dots, x_r be the points in \mathcal{Z}_α which lie over the weight w and correspond to other (finitely many by 1.3.7 [CM98]) roots of $P_\alpha(T)_w \in L[[T]]$, the specialization by w of $P_\alpha(T) \in \mathcal{R}[[T]]$. The $\iota(\alpha)U_{(p)}$ -eigenvalue of x_i ($0 \leq i \leq r$) is denoted by λ_i . By the (local) finite flatness of the weight map \underline{w} (shrinking $\mathrm{Sp} \mathcal{R}$ if necessary) we may assume there are disjoint connected components $\{\mathcal{Z}_i\}_{i=0, \dots, r}$ of \mathcal{Z}_α , such that for any $0 \leq i \leq r$,

- x_i is the only point in \mathcal{Z}_i among the points $\{x_0, \dots, x_r\}$.
- T/λ_0 is topologically unipotent on \mathcal{Z}_0 , and is topologically nilpotent on \mathcal{Z}_i for any $i \geq 1$.
- \mathcal{Z}_i is finite over $\mathrm{Sp} \mathcal{R}$.

Thus $\bigcup_{i=1}^r \mathcal{Z}_i$ is finite flat over $\mathrm{Sp} \mathcal{R}$, hence corresponds to a polynomial $F(T) \in \mathcal{R}[[T]]$ dividing $P_\alpha(T)$. By the construction, we have the following well-defined idempotent operator

$$e = \lim_{n \rightarrow \infty} \left(\frac{\iota(\alpha)U_{(p)}}{\lambda_0} \right)^{n!} \frac{F(\alpha U_{(p)})}{F(\lambda_0)^{-1}},$$

which is easily checked to be the identity on a point in \mathcal{Z}_0 and kill any points in \mathcal{Z}_i for $i \geq 1$.

Consider the integers t such that $Y^{-1}(Y(x) + (p-1)k_0 t) \in \mathrm{Sp} \mathcal{R}$. We form the Siegel-Hilbert modular eigenform of level $H^p p^n$ and weight $\kappa + (p-1)k_0 t$:

$$g_t = e(E^t \cdot f) = e(E^t \cdot g_0).$$

By Proposition 4.5 (applying the first row of diagram (4.3.1) to f) and the continuity of the Hecke action on $M_{H^p p^n, \kappa + Y}^\dagger(\mathcal{R})$, we have that

$$g_t \neq 0, \text{ if } v_p(t) \gg 0.$$

We can write $E^t \cdot g_0$ as a finite sum of classical eigenforms. If f is cuspidal to begin with, then $E^t \cdot g_0$ can be written as a finite sum of cuspidal eigenforms. Pick one of them so that the associated point $x_t \in \mathcal{E}_\kappa$ has image in \mathcal{Z}_0 under the projection r_α . By this construction, the point x is the limit of $r_\alpha(x_t)$ when t goes to 0 p -adically. Now by (4.4.1), one has that x is the limit of x_t . Let f_t be the classical Siegel-Hilbert modular form (recall Remark 4.6) corresponding to x_t . This finally concludes the proof. □

4.5 Complement

In this final subsection we give a complement to theorem 4.15, which is needed for the application [Mo11].

Thus let v be a prime of \mathcal{O} with $v \nmid p$. Fix a Bernstein component \mathcal{B}_v of $\mathrm{GSp}_{2g}(F_v)$. Recall that \mathcal{B}_v is given by the (equivalence class of) data given by a pair (M, τ) , where M is a Levi subgroup of GSp_{2g/F_v} , and τ is a supercuspidal representation of $M(F_v)$, up to twisting by unramified characters of $M(F_v)$. Let E be a number field over which \mathcal{B}_v is defined, and denote by $\mathfrak{z}_v = E[\mathcal{B}_v]$ the affine coordinate ring of \mathcal{B}_v , which is known as the Bernstein centre of \mathcal{B}_v . We have an idempotent element $e_v \in \mathfrak{z}_v$, such that for any irreducible admissible representation π_v of $\mathrm{GSp}_{2g}(F_v)$, we have π_v belongs to the component \mathcal{B}_v if and only if $e_v \cdot \pi_v \neq 0$.

Now we come back to the context of the previous subsections. Let l be the rational prime below the prime v , and let m be the exact power such that l^m divides N . We assume that n is sufficiently large, so that the following holds: denoting by $K_v(m)$ the principal congruence subgroup at the prime v of level n , with associated idempotent $e_{K_v(m)}$. Then we assume that n is large enough so that

$$e_{K_v(m)} \cdot e_v = e_v$$

with e_v being the idempotent associated to the Bernstein component \mathcal{B}_v as above. We may also assume that the extension L/\mathbf{Q}_p in the last subsection to be large enough to contain the number field E .

We note that in the particular case where \mathcal{B}_v is the Iwahori component associated to the (standard) Iwahori subgroup $I_v \subset \mathrm{GSp}_{2g}(F_v)$, then for any π_v belonging to \mathcal{B}_v , we have $U_{(p)}$ acts invertibly on $\pi_v^{I_v}$ (c.f. section 6.4.1 of [BC]).

Back to the fixed Bernstein component \mathcal{B}_v as above. The Bernstein centre \mathfrak{z}_v acts on the space of overconvergent Siegel-Hilbert modular forms $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$. Indeed by the theory of Bernstein centre it suffices to see that the local Hecke algebra of $\mathrm{GSp}_{2g}(F_v)$ with respect to the congruence subgroup $K_v(m)$ acts on $M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$. Since $v \nmid p$, this follows by a similar argument as in section 4.2.

As in chapter 7 of [BC], we can then form the space of overconvergent Siegel-Hilbert modular forms associated to the idempotent e_v :

$$e_v M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R}).$$

Then the same argument as in the proof of theorem 4.15, but with the constructions applied to the space $e_v M_{H^p p^n, \kappa+Y}^\dagger(\mathcal{R})$, yields the following:

Theorem 4.15. *Let f be a classical cuspidal Siegel-Hilbert modular eigenform of weight κ and of level $H^p p^n$. Let π be the cuspidal automorphic representation of $\mathrm{GSp}_{2g}(\mathbf{A}_F)$ generated by f , and assume that π_v belongs to \mathcal{B}_v . There exists, for any positive integer t with $v_p(t)$ large enough, a Siegel-Hilbert cuspidal eigenform f_t of weight $\kappa \cdot \mathrm{Nm}^{(p-1)k_0 t}$ and of the same level, such that the Hecke eigenvalues on the f_t 's converge p -adically to that of f , as $v_p(t) \rightarrow +\infty$. Furthermore the f_t can be taken to have the following property: denoting by π_t the cuspidal automorphic representation of $\mathrm{GSp}_{2g}(\mathbf{A}_F)$ generated by f_t . Then $\pi_{t,v}$ belongs to \mathcal{B}_v .*

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